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CROSS-SECTIONAL DYNAMICS IN A TWO-SIDED MATCHING MODEL

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Cross-Sectional Dynamics in a Two-Sided Matching Model*

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Abstract

This paper studies a dynamic search-theoretic model of two-sided matching with ex ante heterogeneous agents and nontransferable utility. There is a continuum of agents' types in $(0,1)$, with both parties to the match (x,y) receiving flow output xy. Foregone wages are the only cost of search, and all productive relationships are mutual 'tenant-at-will'. Smith (1995a) shows that this belongs to the simple focal subclass of NTU search models in which everyone has the same vNM preferences over matches and thus in steady-state would desire to perfectly segregate. As such, it is an ideal framework to isolate the role of time dynamics, and explore the evolution of long-term relationships.

Despite a continuum of distinct intertwined dynamic optimizations, I characterize the *search equilibrium* starting with everyone unmatched: There is a growing pool of permanently employed on $[\theta_t, 1)$, and an elaborate web of *temporary matches* amongst lower types on $(0, \theta_t)$, where $\theta_t \downarrow 0$. Thus, anyone is eventually is unwilling to accept a temporary match. I develop an equilibrium concept addressing *credibility* constraints, but argue that no matches are lost due to the absence of binding contracts. I relate this to dynamic properties of *flow values* that imply that no one ever quits a match more than once in his life.

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[*] I owe inspiration to a conversation with Doug Galbi, and have especially benefited from the insights of Robert Shimer and Andreu MasColell. The current version also reflects an gradual accumulation of useful comments by Dilip Abreu, Paul Anglin, Roland Benabou, Peter Diamond, Glenn Ellison, Drew Fudenberg, Douglas Gale, Oliver Hart, Bentley MacLeod, Aki Matsui, Thomas Piketty, Peter Sørensen, Jean Tirole, and the participants at the theory workshops at Boston University, MIT/Harvard, Michigan, Montreal, Princeton, Windsor, as well as *Summer in Tel Aviv* 1992 and the 1993 North American Summer Econometric Society Meetings at Boston University. Parts of that paper (or a revision of it) are now in my 1995 MIT mimeo "Matching, Search, and Heterogeneity with Nontransferable Utility." Any errors are mine.

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1. INTRODUCTION

The equilibrium search literature — which dates back to Diamond (1971) — has sought to incorporate transactions frictions into the Walrasian barter process. With remarkably few exceptions, it has proceeded by assuming that the heterogeneity which motivates the search process is entirely or partly realized ex post. That is, after two individuals have met, a stochastic element is realized; only then do they decide upon the wisdom of their potential match or trade. This restrictive 'representive agent' paradigm has provided the wealth of insights we now have about the stylized effect of search frictions on the Walrasian paradigm.

This paper by contrast is part of an overall effort to develop a general theory of search-theoretic matching with ex ante heterogenous agents, and to discover the robust stylized facts of this paradigm, especially as concerns the matching patterns which emerge. Elsewhere in this programme, Shimer and Smith (1995) (hereafter denoted SS) have investigated general transferable utility (TU) models of employment matching in and out of steady-state and attempted to characterize and compare the search equilibria and social optima. Smith (1995a) has focused exclusively on the search equilibria for the nontransferable utility (NTU) paradigm in steady-state. And finally Smith (1995b) has also extended the analysis to the exchange paradigm, and found that search frictions result in 'risk-increasing' trades.

Here, I seek to isolate the specific role of *time,* and how it affects employment matching decisions in the NTU framework. Such a singular focus is made possible by a general insight in Smith (1995a)[1] that in *steady-state,* individuals in these models must *perfectly segregate* into equivalence classes of agents whenever they have *identical von Neumann Morgenstern (vNM) preferences over all potential matches and the unmatched state.* So pure intertemporal considerations prevail to the extent that the matching outcome diverges from this stark scenario. There is a general appreciation by economists that dynamic opportunities can dramatically enrich the range of possible equilibrium outcomes in many environments. Exploring the nonstationary dynamics of this model affords just such a controlled glimpse into the role played by time dynamics in a very fundamental dynamic paradigm.

In this paper, there is a continuum of agents indexed by ability levels in (0,1), where the *flow output* xy of the match (x, y) *is evenly split.* Note that all search environments engendering perfect segregation are essentially isomorphic to this one.[2] Since there is really only 'one side' to the market, this corresponds to what Smith (1995a) has dubbed the *partner model.* I assume that anyone can quit any match at any time, and costlessly search for potential partners who arrive with a known (Poisson arrival) hazard rate. Most crucially, I assume that *all agents are initially unmatched.* The resulting nonstationary dynamics at least partially admits analytic resolution essentially because with identical vNM preferences, a large number of individuals optimally employ *identical threshold strategies.*

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[1] That insight actually first appeared in the (1992) version of this paper.

[2] The first appendix elaborates on this point.

Contributions. My principal contributions are both substantive and methodological. This is (to my knowledge) the first search paper that systematically investigates the cross-sectional *dynamics* occasioned by ex ante heterogeneity. Indeed, the problem is somewhat overwhelming: Any notion of search equilibrium must reconcile a continuum of inherently distinct and yet intertwined individual dynamic optimizations. The identical vNM preferences world thus offers a much-needed foot-in-the-door on this front.

Substantively, this work differs essentially from much of the search literature, for I consider the genesis of long-term relationships. I describe a (conjectured unique) search equilibrium with discounting, and deduce that a growing cohort of *permanently employed*, and an elaborate web of *deliberately temporary matches* beneath them endogenously arises. Now it would be a simple exercise to induce temporary matchings given micro nonstationarities, such as changing preferences or technologies; one contribution of this paper is to highlight and describe the nature of temporary matchings that occur due solely to macro dynamics. This very canonical scenario arises in any newly opened market — be it a 'fraternity rush' or a transforming erstwhile socialist economy, in which there is an implied urgency to act. Indeed, I am effectively forcing just such a rush by assuming that all agents are initially unmatched, and that there is no other force in the system such as entry or match dissolutions that will otherwise counterbalance this effect.[3]

At any rate, I discover an array of interwoven cross-sectional and intertemporal properties of the temporary matches. These I argue capture some of the basic economic issues one must grapple with when pondering the dynamic NTU world. For instance, one might speculate that absent binding contracts, individuals might sometimes be unable to agree upon a mutually beneficial match: For one party might always wish to quit at an unfavourable moment for the other, and this outcome being foreseen in the deterministic world, the match would be declined. That this difficulty does not arise here ensues from the facts that individuals not eligible for permanent employment have decreasing flow unmatched values and are always willing to match with themselves (*self-preference*). As the paper stands, I have not quite finished carefully fleshing out the maze of such interlinked properties.

My major methodological innovation is the general program that I follow, trying to transfer the more easily secured insights about *average* (Bellman) values to deduce what is happening to the *flow* values. There is a rich dynamic and cross-sectional interplay amongst these quantities and the all-important notion of a *reservation wage*. Smith (1994) shows how values and flow values weakly exceed the latter, which happen to be the reservation partner in a world of purely permanent matches, but not in a world of temporary matching. For if one's partner may sometimes quit, the simple monotonic preferences presumed in Smith (1994) no longer obtain, and more sophisticated dynamic analysis is called for. The way in which this inequality plays out is a source of much drama later on in the paper.

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[3] But the TU model of SS — which was not conceived more out of a desire to develop a general theory — *does* have match disolutions, and can illustrate this convergence.

I should also mention that the proof of existence of the nonstationary equilibrium dynamics for permanent employment is also somewhat enlightening, for the dynamical system here is three-dimensional. This precludes the standard application of phase plane diagrams. My proof instead succeeds by an essentially geometric argument that only considers the level surfaces of the system, and effectively reduces the dimensionality of the problem by one. This technique, although not universally applicable, should work in many other unrelated models as well.

Contrast with the Search Literature. Unlike most search models, I assume that output arises *solely* from transferable (or non-specific) human capital. Diamond and Maskin (1979) and (1981), or Wolinsky (1987) are among the many papers in which all agents are ex ante identical, with a random 'match-specific' component at least partially determining the value of the output of the match. But this feint has meant that search models have not been entirely natural extensions of their frictionless Walrasian counterparts.[4] As Smith (1995a) describes, Morgan (1994) has recently offered a different steady-state spin on this framework.[5]

A touchstone of any search model is the agents' margins of decision-making. In this paper, because my goal is to introduce and explore the cross-sectional dynamics, the demands of tractability limit the number of margins of decision afforded the agents. Indeed, previous search papers admit one, three (Sattinger (1992)), or finitely many (in Lockwood's case) ability levels: I have a continuum. Standard search theory is also largely stationary, whereas I explore the nonsteady-state analysis. Thus, unlike Mortensen (1982b), *search intensity* while unmatched is not a choice variable, so that neither agent is deemed 'responsible' for the match. Nor are agents permitted the luxury of searching for alternative opportunities while matched and producing output. Finally, agents do not have recourse to ex post *damages* if partners quit ('breach') their relationships, as in Diamond and Maskin (1979) and (1982a).[6] Rather, all relationships are of the 'tenant-at-will' nature. In my paradigm, the only decision left to the agent is when to agree to a match, and when to quit! Thus, one might imagine there to be no easily verifiable way of ascertaining who broke up a relationship. More plausibly — and the implicit background story in this paper — one could simply suppose that there is no enforcement mechanism for ensuring that any contracted damages are actually paid. Quitting individuals depart 'like a thief in the night.' The result is a tantalizingly simple and surprisingly rich dynamic matching model.

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[4] To draw the distinction more sharply, the heterogeneity in this paper is not only realized ex ante — before the agent matches — but it is long lasting and individual specific. For instance, while the heterogeneity Diamond (1982) was realized in advance of the match, it only lasted for one exchange.

[5] A more established exception is Lockwood (1986), who analyzes the steady-state efficient and equilibrium matching decisions in a model with finitely-many ability levels, where match output is the *sum* of worker ability and a random match-specific component.

I believe that Sattinger (1985) and (1992) are the only previous search-theoretic matching papers with purely ex ante heterogeneity.

[6] Damages were essential there because, unlike here, agents could search while matched.

Within this context, I must decide upon a 'search technology' to employ. There are two polar canonical choices:[7] With the *linear* technology, finding partners is no easier with more searchers than with less, while the ease of matching is linearly proportional to the number of searchers with the *quadratic* technology. In keeping with my intention to isolate the effect of time alone, I wish to avoid complicating congestion effects that are better analyzed elsewhere. I thus work exclusively with the quadratic technology, due to a decision-theoretic advantage that I make patently clear: That is, the matching decisions of agents are *separable* from those of agents with whom they do not plan to match. This knife-edge analytic property renders potentially horrendous equilibrium calculations rather pedestrian — for this allows agents planning permanent matches to derive their common equilibrium strategy independent of those engaged in temporary matches.

Outline. The next section briefly analyzes the competitive equilibrium and first-best matching without search frictions. Section 3 is entirely self-contained, and provides a flavour of the equilibrium and stylized predictions in analogous two and three period versions of the model. Section 4 carefully presents the continuous-time model, and some basic tools are then developed in section 5. What follows is an analysis of the permanent matching that arises in section 6, and the temporary matching in section in section 7. The existence proofs are as yet not included, awaiting some final innovations in SS. Some concluding remarks and an appendix containing the less economic proofs follow.

2. COMPETITIVE EQUILIBRIUM WITHOUT SEARCH

Suppose there is a continuum of individuals with efficiency parameters distributed uniformly in (0,1). If individuals x and y are matched, they produce a flow product of xy. In light of the Second Welfare Theorem, I shall try to exhibit a first-best static matching, and try to decentralize it. That is, what is the matching that maximizes total flow output here?[8] Rather than appeal to optimal control theory for a rigorous argument, simply consider the following heuristic instead. An interchange lemma due to Hardy asserts that for any two positive increasing sequences $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, the maximum of the expression $\Sigma_{i=1}^{n} a_i b_{\pi(i)}$ over permutations π of $\{1, 2, \ldots, n\}$ is the identity permutation: Pair the highest ι's with the highest b's. For my scenario, this suggests that each $x \in (0,1)$ should be paired with another x, yielding an average output of $\int_0^1 x^2 dx = 1/3$.

I now describe the competitive equilibrium of the model.[9] That is, I temporarily consider a world with truly variable wages — i.e. not necessarily equal output splits. I first claim that there is a competitive equilibrium in which x is always paired with x and earns a wage equal to $x^2/2$. For just notice that any x can clearly afford to make such a wage offer to another x, while if x were to meet $y \neq x$, then wages would over-exhaust output, as

---

[7] But see SS for a generalization.

[8] This simple question has the flavour of the asymmetric optimal pairing question studied by Kremer and Maskin (1994).

[9] This remark was suggested to me by Roland Benabou.

$$xy - x^2/2 - y^2/2 = -(x-y)^2/2 < 0.$$

The same logic also proves that such an equilibrium is unique.

3. DISCRETE-TIME NONSTATIONARY SEARCH: EXAMPLES

How closely can optimal selection by individuals in a two-sided matching model approximate this socially optimal frictionless outcome? That is, suppose that this particular market is newly opened, and that all individuals are initially unmatched. Let the per period discount rate be $\delta < 1$. Consider the following two distinct paradigms:

1. Matches are proposed to every unmatched agent in each period;
2. In all but the last period, each unmatched agent has a match proposed to her with probability equal to the unemployment rate. In the final period, all remaining unpaired are randomly matched.

In each paradigm, either party to a proposed match is permitted a veto. Output is equally split. Further assume that search is a time-consuming process, so that if one is matched in a given period, one cannot simply quit that match for another the very next period; rather, a one period sabbatical is required to search. With or without discounting, one avenue for resolving this model is to consider the following finite-horizon truncation: Let the matching process extend T periods, with the period-T payoff equal to the flow reward divided by $(1-\delta)$. Equivalently, all unmatched agents after period T are randomly paired forever, with no further quits or vetoes permitted; payoffs, however, accrue over the infinite horizon. Figure 1 outlines the timing and payoffs.

Figure 1: **Discrete-Time Truncated Model.**

period	$t = 1, 2, \ldots, T-1$	T
activity	matched $\Longrightarrow$ produce x_t (then quit?) unmatched $\Longrightarrow$ search again (then accept?)	produce x_T
payoff	$\delta^t x_t$ if matched; else 0	$\delta^T x_T/(1-\delta)$

It is a sorry fact that there is no closed form solution of the infinite horizon problem in discrete time. Still, much of the economic insight is found in the two- and three-period truncations, to which I now turn.

3.1 A Two Period Example

Let there be two periods in which search can occur. Here there is no distinction between the two matching paradigms. In the first period, each individual randomly meets another potential partner. If either vetoes the match, each receives zero payoff that period. Otherwise, they split the output produced equally. In the second period, all individuals remaining unmatched are randomly paired, once for all. No vetoes are permitted.[10] Clearly, no individual will ever opt to terminate a match. Thus, individuals effectively have two strategies: match or don't match in the first period.

Note that the first period decision criterion of all individuals is identical: Given the multiplicatively separable production function, one's own parameter has no effect on preferences. Each individual simply seeks to maximize the discounted expected parameter of her partner. Thus, in period one, all individuals will accept any parameter above some threshold θ equal to their discounted expected period two partner's parameter. Given that everyone uses the same threshold,

$$\theta = \delta \frac{\theta(\theta/2) + (1-\theta)\theta(1+\theta)/2}{\theta + (1-\theta)\theta}$$

so that $(2-\delta)\theta^2 - (4-\delta)\theta + \delta = 0$ or $\theta = 0$. There is also an equilibrium with $\theta = 0$ that is possible: All individuals who meet in a given period agree to match, knowing that no one will be around next period. This *trivial* outcome is excluded from consideration.[11] Thus,

$$\theta = \frac{4 - \delta - \sqrt{5\delta^2 - 16\delta + 16}}{2(2-\delta)},$$

and there is aequivalence class $[\theta, 1)$ of individuals willing to pair with one another in the first period, and no one with an efficiency parameter below θ will pair until period two.

It can be shown that the earnings profile is discontinuous at θ. Individuals with indices above θ earn rents, due to the equal output division rule. This creates an unusual desire for agents just below θ to prefer small increases in the interest rate (more heavy discounting of future payoffs) because θ is increasing in δ.

3.2 A Three Period Example

Now suppose that there are three periods, with those matches proposed in the final period necessarily consummated. Individuals now enjoy four strategies: (*i*) match in period one (if given the opportunity) and stay matched; (*ii*) match, and then quit; (*iii*) refuse the first period match and then match; and finally (*iv*) refuse the matches offered in the first two periods. I shall soon rule out the last strategy.

---

[10] Or, vetoes are allowed, but would never be used since the alternative payoff equals zero.

[11] One can also argue that it is not stable either, in the sense that it collapses if some individuals "tremble," and reject their first period partner.

I proceed by backward induction. Independent of non-trivial first period behaviour, the optimal second period strategy is to accept a proposed partner exactly when her index is at least $\theta_2 > 0$, where θ_2 *equals the discounted expected period three partner's parameter.*[12] Thus, there is some equivalence class $(\theta_2, 1)$ in period two of mutually desirable individuals. Moreover, by the rationale of the previous example, no individual will ever opt to terminate a match after this period.

Next consider the first period behaviour. Since θ_2 is the lower threshold for accepting any *new* matches in period two, anyone initially paired in period one will quit her match in period two (and thus forego period two payoffs) if and only if her partner's parameter lies below θ_2. Who in the first period then will agree to match? Clearly, an individual declines a match exactly when her discounted expected payoff from being *eligible* for next period's matching exceeds that of the proposed match. Since those with indices below θ_2 will always be turned down (or dumped, if already matched) in the second period, the option of being eligible for the next period is worthless. They will therefore accept any proposed match in the first period. (I have now ruled out strategy (*iv*) above.)

Consider now those with indices at least θ_2. Since they are all in the same equivalence class in the second period, they all have the same first period threshold θ_1. Note that θ_1 *equals the discounted expected eventual partner's parameter, conditioned on* θ_2 — since these matches are not quit. Then

Lemma 1 $\theta_1 > \theta_2$

Proof: Denote by $P_t(a, b)$ the fraction of eligible individuals at the outset of period t with efficiency parameters y in (a, b), and by E_t the expectation operator with respect to the measure P_t. Then $\theta_1 = P_2(0, \theta_2)\theta_2 + P_2(\theta_2, 1)E_2(y \mid y \geq \theta_2)$, and so

$$\begin{aligned}
(\theta_1 - \theta_2)/\delta &= [P_2(0, \theta_2)\theta_2 + P_2(\theta_2, 1)E_2(y \mid y \geq \theta_2)] - E_3(y) \\
&= [P_2(0, \theta_2)\theta_2 + P_2(\theta_2, 1)E_2(y \mid y \geq \theta_2)] - \\
&\quad [P_3(0, \theta_2)E_3(y \mid y < \theta_2) + P_3(\theta_2, 1)E_3(y \mid y \geq \theta_2)] \\
&= P_2(0, \theta_2)\theta_2 + [P_2(\theta_2, 1) - P_3(\theta_2, 1)]E_2(y \mid y \geq \theta_2)] - \\
&\quad P_3(0, \theta_2)E_3(y \mid y < \theta_2) \\
&> P_2(0, \theta_2)\theta_2 + [P_2(\theta_2, 1) - P_3(\theta_2, 1)]\theta_2 - P_3(0, \theta_2)\theta_2 = 0
\end{aligned}$$

because the second period matching equivalence class implies that in the first paradigm

$$P_3(\theta_2, 1) = \frac{P_2(\theta_2, 1) - P_2(\theta_2, 1)^2}{1 - P_2(\theta_2, 1)^2} = \frac{P_2(\theta_2, 1)}{1 + P_2(\theta_2, 1)} < P_2(\theta_2, 1)$$

while in the second paradigm, $P_3(\theta_2, 1) = P_2(\theta_2, 1) - P_2(\theta_2, 1)^2 < P_2(\theta_2, 1)$ easily follows. ◇

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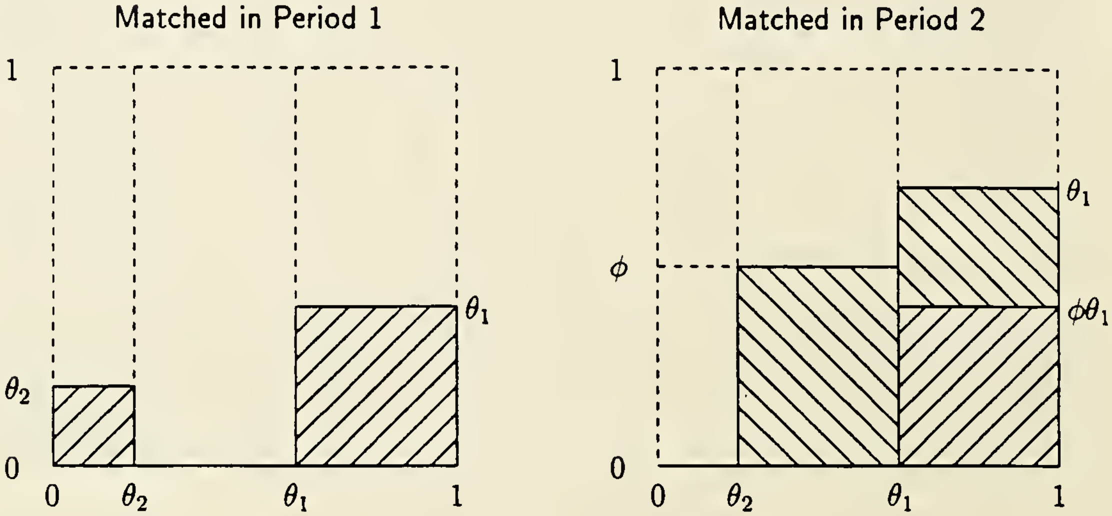
[12] This expectation depends in an obvious fashion upon the common period two behaviour. Thus, whether such a θ_2 exists or is unique is at issue. I shall, however, shortly exhibit a solution which happens to be unique.

Figure 2: **Histogram of Matched Types in the Three Period Model.** Shaded in this figure are the measures of agents of each type who are matched in the first two periods.



Given $\theta_1 > \theta_2$, the *histogram* diagram of figure 2 — with agents' parameters on the horizontal axis and fractions matched equal to the shaded height — depicts the matches that occur. A few stylized facts emerge from the above.

1. In period one, the very best and the very worst agree to match: The two (forward shaded) equivalence classes are $(0, \theta_2)$ and $[\theta_1, 1)$; the former consists of purely *temporary matches*;
2. *Quits* occur: It is common knowledge that matches consummated within $(0, \theta_2)$ are only temporary, with those individuals ineligible for second period matches;
3. Among those not seeking permanent matches, a partial "Groucho Marx result" manifests itself: In the first period, no one in (θ_2, θ_1) is willing to match with anyone who's willing to match with her — namely those in $(0, \theta_2)$.
4. As time progresses, there is less self-segregation: More of the very best and fewer of the very worst agree to match: In period two, there is one such (backward shaded) equivalence class $[\theta_2, 1)$, and in the final period the equivalence class constitutes all of $(0, 1)$.[13]

Having developed some simple intuitions in discrete time, to push the model much further, I shall switch to continuous time; this will permit a more crafted analytic discussion of the nature of equilibrium. Observations 1, 2, and 4 will prove robust in the continuous time infinite horizon context, while investigating the generality of observation 3 will prove pivotal for the analysis of temporary matching in the continuous time model.

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[13] See the Appendix A.2 for a derivation of these thresholds.

4. THE CONTINUOUS TIME MODEL

4.1 Agents

Agents are indexed by their *type* (or *efficiency parameter*) $x \in \mathbb{R}$. Let λ be the time-invariant Borel measure on types, with support $\Sigma = (\underline{x}, 1)$, for some $\underline{x} \in [0, 1)$. I shall normalize the measure of everyone to unity, so that $\lambda(\mathbb{R}) = \lambda(\Sigma) = 1$.[14] So as to avoid the complications occasioned by an atomic type distribution, I simply assume that λ is differentiable with respect to Lebesgue measure, with Radon-Nikodym derivative ℓ (a.e.). I shall let L be the associated c.d.f.

It may help to imagine that individuals actually belong to $\mathbb{R}^2$, with a continuum of agents of each type x having (one-dimensional) measure $\ell(x)$. In that case, I may still without loss of generality simply refer to an agent by her type x as I implicitly show that one's optimal strategy is solely a function of one's type. For instance, in the earlier histograms, λ was Lebesgue measure on $(0,1)$, and $\ell(x) \equiv 1$.

4.2 Action Sets

At any instant in continuous time, an individual is either *matched* (*employed*) or *unmatched* (*unemployed*). Only the unmatched engage in (costless) search for a new partner.[15] When two agents meet, their efficiency parameters are *perfectly observable* to each other. Matches can be thought of as *mutual tenant-at-will*: Either party may veto the proposed match; if both approve, it is consummated and remains so either until one partner quits. The equilibrium concept shall rule out 'imperfect' behaviour, in which one player vetoes because she anticipates that the other will; such strategic considerations are simply not the focus of this paper. At the moment the match is severed, both individuals instantaneously re-enter the pool of searchers.

4.3 Match Payoffs

For now, assume the production technology is bilinear: If x is paired with y then the flow output of that relationship is simply the product xy — which is equally divided. Together, the above assumptions on joint production and output division allow me to normalize players' payoffs by their own indices, and redefine the *flow payoff* at time t of each individual x to be her partner's index $p_t(x)$. If x is unmatched at that moment, let $p_t(x) = 0$.

Turning to the agents' infinite-horizon objective function, everyone discounts future payoffs at the common *interest rate* $\beta > 0$. Thus, each x maximizes her expected infinite-horizon discounted average payoff $E[\beta \int_0^\infty e^{-\beta t} p_t(x) dt]$.

---

[14] So long as $\int_0^\infty x\lambda(dx) < \infty$, all results of this paper obtain for $\text{supp}(\lambda) = [0, \infty)$. But the proofs are more tedious, and so this minor piece of generality is avoided.

[15] Neither search costs nor on-the-job search enrich the analysis so much as complicate it, at least for the few points that I wish to make.

4.4 The Measures of Matched and Unmatched Agents

At any time t, I must be able to precisely describe the distribution of existing matches, which will turn out to be the *state variable* for the model. Closely following SS, let $\mu_t(X,Y)$ be the mass of agents with types in $X \subset \Sigma$ who are matched with agents with types in $Y \subset \Sigma$. Since obviously $\mu_t(X,Y) \equiv \mu_t(Y,X)$, each *matching measure* μ_t lies in $\mathcal{M}^s$, the space of nonnegative measures on Σ^2 that are uniformly bounded above by 1 and are *symmetric* on $\Sigma \times \Sigma$. The resulting measure-valued function $\langle \mu_t \rangle : [0,\infty) \rightarrow \mathcal{M}^s$ is absolutely continuous with respect to $\lambda^2 \equiv \lambda \times \lambda$, the measure of agents with types in the pair of sets (so that $\lambda^2(X,Y) = \lambda_t(X)\lambda_t(Y)$) — so let $m_t \equiv d\mu_t/d\lambda^2$ be the Radon-Nikodym derivative. Hence,

$$\mu_t(X,Y) = \int_X \int_Y m_t(x,y)\,\lambda(dy)\,\lambda(dx)$$

Here, $m_t : \mathbb{R}^2 \mapsto \mathbb{R}_+$ is the *match density function*, and $m_t(x,y)$ may be interpreted as the normalized density of existing matches (x,y), where the normalization accounts for the frequency of the types x and y in the population. Note that $m_t(x,y) \equiv m_t(y,x)$, for λ^2-a.e.-(x,y), since μ_t is symmetric.

I shall let $\nu_t(X)$ denote the mass of agents with types in $X \subset \Sigma$ that are unmatched at time t. This *unemployment measure* satisfies

$$\nu_t(X) = \lambda(X) - \mu_t(X,\Sigma) \equiv \lambda(X) - \int_\Sigma \int_X m_t(x,y)\,\lambda(dx)\lambda(dy). \tag{1}$$

Once again, if $u_t \equiv d\nu_t/d\lambda$ denotes the Radon-Nikodym derivative, then $\nu_t(X) \equiv \int_X u_t(x)\lambda(dx)$. Then (1) implies that $u_t(x) = 1 - \int_\Sigma m_t(x,y)\,\lambda(dy)$ is the *point unemployment rate* of type x. The overall unemployment rate (and mass, since $\lambda(\Sigma) = 1$) in the economy is denoted $u_t = \nu_t(\Sigma) \equiv 1 - \int_\Sigma \int_\Sigma m_t(x,y)\,\lambda(dy)\,\lambda(dx)$.

4.5 The Search Technology

'Linear' and 'Quadratic' Cases. With search frictions, match creation takes time. I wish to use the simplest possible search technology involving Poisson arrival times. In the terminology of SS, I shall only consider *anonymous* search technologies, whereby individuals meet one another in direct proportion to their mass in the unmatched pool. But what then is the constant of proportionality? In a *linear search technology,* potential partners for unmatched individuals arrive with constant flow probability (or 'hazard rate') $\rho > 0$, independent of the unemployment rate. Thus, an unmatched individual can expect to have a match proposed to her within time $t > 0$ with probability $1 - e^{-\rho t}$. Call ρ the *rendezvous rate.*

By contrast, in the *quadratic search technology,*[16] the arrival rate of potential partners for unmatched individuals equals ρ *times the unemployment rate.* Put differently, *were it possible,* an unemployed individual *would* meet another unemployed or employed partner according to a Poisson process with arrival rate ρ. But as it is presumed technically infeasible to match with someone who is already employed, the overall arrival rate of potential partners equals ρu_t. Simply put, the rate at which a searcher meets any subset of individuals is directly proportional to the *measure* of searchers in that subset.[17]

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[16] To motivate these labels, taken from Diamond and Maskin (1979) and (1981), observe that in the linear (resp. quadratic) technology, the total flow of meetings is proportional to the unemployment rate (resp. its square).

Strategic Separability. The analysis of this paper shall be confined to the quadratic technology in light of a crucial analytic knife-edge advantage. Consider the following slightly altered story: *Invitations to possible meetings arrive at* fixed *rate* ρ, *but now with a chance* $1 - u$ *the other individual is already paired, and thus misses the meeting.* So call the quadratic matching technology *strategically separable,* since an individual only cares about the arrival rate of acceptable agents, and with the quadratic technology (only), this is purely a function of their measure. That is, her *current* match payoffs are unaffected by any matching decisions made by those with whom she is unwilling to match. (Whether she is *permanently* unaffected is another matter entirely that this paper must grapple with.) While such an absence of 'crowding out', or congestion effects, has been conveniently exploited in the past, it is absolutely essential here in this nonstationary environment.

4.6 Strategies

I make the major simplifying assumption that strategies are dependent only on payoff-relevant state variables, and are not otherwise conditioned on history. This will be my justification for analyzing the model purely in aggregate distributional terms. In this model, the sequence of measures $\langle \mu_t \rangle$ is the only payoff-relevant state variable. Because $\langle \mu_t \rangle$ evolves deterministically without aggregate uncertainty, it therefore suffices to simply condition strategies on time alone.

Pure Strategies. At any moment in time, an unmatched individual must choose which matches to agree to should a meeting occur, whereas someone already matched must decide which matches are worth continuing. I shall suppose that these two sets coincide — a restriction which is innocuous in the absence of fixed costs of forming or dissolving matches, and not tenable otherwise.

Hence, pure strategies can be represented simply by the graph of the *acceptance correspondence* $\mathcal{A}_{\cdot}(x) : \mathbb{R}_+ \rightrightarrows \Sigma$, with time on the horizontal axis, and types x on the vertical axis. So the *acceptance set* $\mathcal{A}_t \subseteq \Sigma$ specifies whom x is willing to be matched with at time t, and is assumed *right* (upper and lower hemi-) continuous in time; this rules out a large but uninteresting class of weakly dominated strategies.

A possibly more appealing representation of strategies takes horizontal rather than vertical slices of $\mathcal{A}$. Define the *quitting time* $\tau_t(x, y) = \sup\{s \geq t \mid y \in \mathcal{A}_{s'}(x)$

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[17] It is worth remarking that paradigms 1 and 2 in the discrete-time examples correspond to the linear and quadratic technologies, respectively, with $\beta = -\log \alpha$ and $\rho = 1$ (if the real time length of a period was 1).

for all $s' \in [t, s)\}$, i.e. the first time after t that x plans to quit her match with y, given that she is, or has the opportunity to be, matched with y at time t. In concordance with my assumption about acceptance sets, I assume that $\tau_t(\cdot,\cdot)$ is right continuous in time t.[18]

Mixed Strategies. The space of time-0 mixtures over acceptance correspondences or stopping times $\mathcal{T}$ is simply too intractable to represent dynamics.[19] Fortunately there is a simpler restricted mixed strategy space with many nice properties. The resulting mixtures are analogous to behaviour strategies in game theory, insofar as they instruct players how to behave at each moment in time. A *Markovian quitting rule* specifies a chance $Q_s(t; x, y)$ that x does not quit the match (x, y) by time t given that at time $s \leq t$, either (i) the match already obtains or (ii) a meeting occurs, and given that the match is not otherwise dissolved in (s, t). This dual usage reflects some of the cutting power of the Markovian spirit.[20] By analogy to the properties of pure strategies, assume that $Q_s(t; x, y)$ is right continuous in $t \geq s$. SS show that $Q_s(t; x, y)$ is weakly monotonically decreasing in $t \geq s$, for fixed s, and that it satisfies the *Bayes-Markov property*: $Q_{t_0}(t_2; x, y) = Q_{t_0}(t_1; x, y) Q_{t_1}(t_2; x, y)$, for $t_0 < t_1 < t_2$.

4.7 Distributional Dynamics

Joint Quitting Chances. Markovian quitting rules by both parties determine the rate at which any given match is being destroyed. Intuitively, since players' randomizations are generally assumed to be independent in equilibrium,[21] $P_s(t; x, y) \equiv Q_s(t; x, y) Q_s(t; y, x)$ is the *joint quitting probability.* That is, this is the symmetric chance that neither party quits the match (x, y) by time t given that at time $s \leq t$, either the match already obtains or a meeting occurs.

Much of the analysis demands that I essentially gloss over the behaviour of individual types in the model, and instead focus on positive masses of agents. So let $\Pi_s(t; X, Y)$ be the fraction of matches $(x, y) \in (X, Y)$ that are not quit by either party by time t given that the match already obtains at time $s \leq t$. Then $\Pi_s(t)$ is absolutely continuous with respect to μ_s, with Radon-Nikodym derivative $P_s(t; x, y)$ at (x, y). Consequently,

$$\Pi_s(t; X, Y) = \int_X \int_Y P_s(t; x, y) \mu_s(dx, dy) = \int_X \int_Y P_s(t; x, y) m_s(x, y) \lambda(dx) \lambda(dy)$$

Stocks and Flows. Match creation and quits determine the time path of $\langle \mu_t \rangle$, and thus of $\langle m_t(x, y) \rangle$. Since all strategies are right continuous in time, so also are $\langle \mu_t \rangle$. To begin with, I assume that all individuals are unmatched, so that $\mu_0 \equiv 0$. Then because current matches are just those that have been formed at some time in the past and have not been destroyed in the interim, I have

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[18] I should note in passing that quitting times τ must obey a simple *dynamic consistency* requirement: If $\tau_t(x, y) > t$ then $\tau_s(x, y) \equiv \tau_t(x, y)$ for all $s \in [t, \tau_t(x, y))$.

[19] As pointed out in Salinetti and Wets (1981), it is possible to nicely topologize this space using the Vietoris finite topology.

[20] Appendix A.3 discusses a nice strategic restriction embodied in such quitting rules.

[21] Appendix A.4 addresses concerns raised by this assumption.

$$\mu_t(X,Y) = \int_0^t \rho\nu_s(X,Y)\Pi_s(t;X,Y)ds \tag{2}$$

Note that $\Pi_s(t)$ determines the 'flows' into and out of the current 'stock' μ_t of matches. To prove the existence of equilibrium later on, I must eliminate any reference to the flows. This will be possible because SS shows that for any $t > s$,

$$\Pi_s(t;X,Y) = e^{\int_s^t \delta - \rho\nu_{s'}(X,Y)/\mu_{s'}(X,Y)ds'}\mu_t(X,Y)/\mu_s(X,Y) \tag{3}$$

SS also proves that the matching measure $\langle \hat{\mu}_t \rangle$ belongs to the space $D^f_{\mathcal{M}^s}[0,\infty)$ of right-continuous functions $[0,\infty) \to \mathcal{M}^s$ with left hand limits, meeting the feasibility conditions

$$0 \leq \mu_t(X,Y) \leq \mu_s(X,Y)e^{\int_s^t \rho\nu_{s'}(X,Y)/\mu_{s'}(X,Y)ds'} \tag{4a}$$

$$\dot{\mu}_t(X,Y) \leq \rho\nu_t(X,Y) \tag{4b}$$

5. AGREEABLE MATCHES AND SEARCH EQUILIBRIUM

5.1 Value Equations

Let $\mathcal{W}_t(x)$ be the expected *average present value* for x of remaining unmatched at time t and thereafter behaving optimally. Let $\mathcal{W}_t(x|\, y)$ be the corresponding average value to x of either matching with y, or having the opportunity to do so, at time t. So $\mathcal{W}_t(x|\, y) \geq \mathcal{W}_t(x)$. There is a natural relationship between these two quantities: An unmatched individual x receives a zero payoff until her first potential match y arrives at time s, whereupon she gets an average utility $\mathcal{W}_s(x|y)$, discounted back to the present. Hence,

$$\mathcal{W}_t(x) = \int_t^\infty e^{-\int_t^s \left(\beta + \int_\Sigma \rho u_{s'}(y)\,\lambda(dy)\right)ds'} \cdot \int_\Sigma \rho u_s(y)\mathcal{W}_s(x|y)\,\lambda(dy)\,ds \tag{5}$$

Just as in SS, corresponding to the *average* value $\mathcal{W}_t(x)$, I shall define the *flow value* $\psi_t(x) = \mathcal{W}_t(x) - \dot{\mathcal{W}}_t(x)/\beta$. This turns out to be the *instantaneous reservation partner* of x at time t: the type with whom x is indifferent between remaining unemployed and matching for an arbitrarily short period of time, during which she cannot search. Solving the differential equation for $\mathcal{W}_t(x)$ produces the implicit equation $\mathcal{W}_t(x) \equiv \int_t^\infty \beta e^{-\beta(s-t)}\psi_s(x)\,ds$. This theoretical construct plays a key role in much of my analysis. Indeed, differentiation of (5) yields

$$\psi_t(x) = (\rho/\beta)\int_\Sigma u_t(y)\,(\mathcal{W}_t(x|y) - \mathcal{W}_t(x))\;\lambda(dy) \tag{6}$$

Results in SS imply that the value to x of the match with y is

$$\mathcal{W}_t(x|y) = \int_t^\infty \beta e^{-\beta(s-t)} \left(P_t(s;x,y)y + (1 - P_t(s;x,y))\,\psi_s(x)\right) ds \tag{7}$$

The average value to x of a match with y at time t is the discounted average of flow playoff y, weighted by the probability the match *has not* been dissolved by time s, plus the discounted average of the future (hypothetical) instantaneous reservation partner $\psi_s(x)$, weighted by the probability the match *has* dissolved by time s. Next substitute $\psi_s(x) = \beta\mathcal{W}_s(x) - \dot{\mathcal{W}}_s(x)$ into the above to find that the *own-surplus* $\mathcal{S}_t(x|y) \equiv \mathcal{W}_t(x|y) - \mathcal{W}_t(x)$ satisfies:

$$\mathcal{S}_t(x|y) = \int_t^\infty e^{-\beta(s-t)}\; P_t(s;x,y) \left(\beta\,(y - \mathcal{W}_s(x)) + \dot{\mathcal{W}}_s(x)\right) ds \tag{8a}$$

$$= \int_t^\infty \beta e^{-\beta(s-t)} P_t(s;x,y)(y - \psi_s(x))\, ds \tag{8b}$$

Now, while unmatched for an instant, x foregoes the ephemeral flow chance of meeting an individual such as y who would provide her with own-surplus $\mathcal{S}_t(x|y)$. Substituting (8b) into (6) thus yields the following accounting equation:

$$\psi_t(x) = (\rho/\beta) \int_\Sigma u_t(y) \left(\int_t^\infty \beta e^{-\beta(s-t)} P_t(s;x,y)(y - \psi_s(x))\, ds\right) \lambda(dy) \tag{9}$$

So the flow value of x is increasing in both her expected rendezvous probability and the expected surplus she will derive from future matches over her unmatched value. The derivation of this implicit flow value equation is much more direct here than in the TU model of SS, who must first consider the Nash bargaining solution.

I also must consider the twin notion of a *permanent reservation* or *threshold partner* $\theta_t(x)$ for type x. By this, I mean the type with whom x is indifferent about matching — *assuming that that individual will never quit the match.* Thus, if x is willing to match with y, then $y \geq \theta_t(x)$, while conversely, if $y \geq \psi_t(x)$ then x is willing to match with y. The theory describing the relationship between $\mathcal{W}_t(x)$, $\psi_t(x)$, and $\theta_t(x)$ is developed in Smith (1994). To summarize, all three coincide in steady-state. But in general, $\theta_t(x) \leq \min\langle \mathcal{W}_t(x), \psi_t(x)\rangle$, because there is an 'option to renew' bundled with $\theta_t(x)$. Moreover, $\theta_t(x) = \mathcal{W}_t(x) < \psi_t(x)$ when this option is forever exercised, as when $\theta_t(x)$ or $\mathcal{W}_t(x)$ is at a strict global maximum on $[t, \infty)$; conversely, $\theta_t(x) = \psi_t(x) < \mathcal{W}_t(x)$ exactly when this option is not worthless, because $\theta_t(x)$ is locally increasing. In the intermediate case, I have $\theta_t(x) < \min\langle \mathcal{W}_t(x), \psi_t(x)\rangle$, with no general ordering among $\mathcal{W}_t(x)$ and $\psi_t(x)$.

5.2 Which Matches are Mutually Agreeable?

I now identify the sets of mutually agreeable matches, which shall be the basis for the equilibrium notion. Intuitively, a match (x, y) is *mutually agreeable* at time t if $(\star)$ holds: $\mathcal{W}_t(x|\,y) > \mathcal{W}_t(x)$ and $\mathcal{W}_t(y|\,x) > \mathcal{W}_t(y)$, or equivalently $\mathcal{S}_t(x|y) > 0$ and $\mathcal{S}_t(y|x) > 0$. Later on, I may wish to further nuance and call

(x, y) *weakly mutually agreeable* if $(x, y) = \lim_{n\to\infty}(x_n, y_n)$ for mutually agreeable matches $\{(x_n, y_n)\}$. But I really need a reformulation in terms of the more primitive unmatched values, and this leads to an interesting discovery.

Observe that for (x, y) to be mutually agreeable, it is *necessary* that $y > \theta_t(x)$ and $x > \theta_t(y)$. But this is by no means *sufficient* because it is crucial that there exist some *common* time interval $[t, s)$ that both individuals wish to remain matched for, and this restriction is not reflected in that assertion. Fortunately, Smith (1994) proves that $y > \theta_t(x)$ iff $\mathcal{W}_t(x) < y + e^{-\beta(s-t)}(\mathcal{W}_s(x) - y)$ for some $s > 0$. So, reformulating $(\star)$, the more primitive assertion about average values requires that there exist some $s > t$, such that

$$\mathcal{W}_t(x) < y + e^{-\beta(s-t)}(\mathcal{W}_s(x) - y) \tag{10a}$$

$$\mathcal{W}_t(y) < x + e^{-\beta(s-t)}(\mathcal{W}_s(y) - x) \tag{10b}$$

It so happens that even this dual condition is not sufficient.[22] Indeed, one player, say x, could conceivably wish to quit at a previous time, rendering the match on balance ex ante strictly unprofitable for y; or if not, y may then wish to quit at an even earlier time, etc. (See figure 3.) Say that a match (x, y) has *contractable surplus* if it satisfies (10) for some time $s > t$. One could simply posit that binding contracts exist as a remedy for any such self-imploding yet matches with contractable surplus; however, it is not at all obvious *which* outcome would be agreed upon, as the different parties have different preferences. Resolving this conflict would likely entail side payments, and this would allow one to circumvent the NTU model entirely — for individuals would then have an incentive to maximize the joint surplus, as in the TU model.

I shall therefore respect the mutual tenant-at-will paradigm, and carefully formulate the correct notion of mutual agreeability. Denote by $\mathcal{S}_t(s; x, y)$ the *running own-surplus* to type x if her match with y lasts from time t until time s. Then

$$\begin{aligned}\mathcal{S}_t(s; x, y) &\equiv (1 - e^{-\beta(s-t)})y + e^{-\beta(s-t)}\mathcal{W}_s(x) - \mathcal{W}_t(x)\\ &= \int_t^s \beta e^{-\beta(r-t)}(y - \psi_r(x))\,dr\end{aligned}$$

I now formulate mutually agreeability indirectly (almost) in terms of primitives. It requires that both iterated incentive compatibility constraints and individual rationality be met.

Lemma 2 (Mutually Agreeable Matches) *At time t, the match (x, y) is mutually agreeable until time $\hat{s} > t$ exactly when*

- [IC] $\hat{s} = s_n = s_{n-1}$ *where the sequence* $\langle s_k\rangle$ *satisfies* $s_0 = \infty$ *and* $s_{k+1} = \min\langle \arg\sup_{t\le s<s_k} \mathcal{S}_t(s; x, y), \arg\sup_{t\le s<s_k} \mathcal{S}_t(s; y, x)\rangle$ *for all* k*;*
- [IR] $\mathcal{S}_t(\hat{s}; x, y) \geq 0$ *and* $\mathcal{S}_t(\hat{s}; y, x) \geq 0$ *with strict inequality for at least one.*

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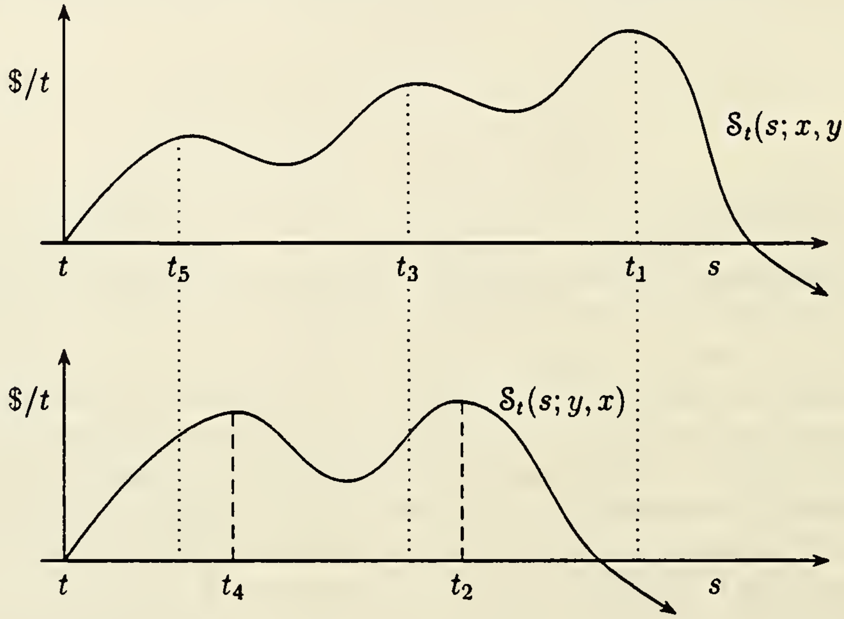
[22] Many thanks to Dilip Abreu for alerting me to this nicety.

Figure 3: **How Tenant-at-Will Incentive Compatibility Constraints Can Unravel a Potential Match with Contractable Surplus.** From the two graphs of running own-surplus, it is apparent that x ideally would wish the match to last until time t_1, but y prefers to quit at time $t_2 < t_1$. Rationally foreseeing this, x would then prefer to quit at time $t_3 < t_2$. But then y would wish to quit at time $t_4 < t_3$. The mutually agreeable outcome subject to incentive compatibility is for the match (x, y) to last until time t_1. Had it been the case that $\mathcal{S}_t(s; x, y) < 0$ until time $(t_4 + t_5)/2$, the match would not be mutually agreeable.



If the IR constraint binds for both parties (and yet $\hat{s} > 0$), simply call the match *weakly mutually agreeable.* All other matches are *mutually disagreeable.* Let $\mathcal{MA}_t$ be the set of all mutually agreeable matches (x, y) at time t, and let $\mathcal{MA}_t(x) = \{y \mid (x, y) \in \mathcal{MA}_t\}$. Similarly define $\mathcal{MD}_t$ and $\mathcal{MD}_t(x)$ for mutually disagreeable matches (x, y) at time t.

5.3 Search Equilibrium

I now adapt the notion of *search equilibrium* developed for the purely TU model of SS. In a word, I demand that all mutually agreeable matches $(x, y) \in (X, Y) \subseteq \mathcal{MA}_t$ be maintained and all mutually disagreeable matches $(x, y) \in (X, Y) \subseteq \mathcal{MD}_t$ be quit at once. In order to establish existence of equilibrium, I shall have to express this notion wholly in terms of primitives. To this end, notice that $\mathcal{S}_t(s; x, y)$ is decreasing in s exactly when her then *flow surplus* $y - \psi_s(x) < 0$. Hence, search equilibrium will ask that a match continue so long as both individuals have positive 'continuation surplus':

$$\min\left\langle \int_t^\infty e^{-\beta(s-t)} P_t(s;x,y)(y-\psi_s(x))\,ds, \int_t^\infty e^{-\beta(s-t)} P_t(s;x,y)(x-\psi_s(y))\,ds \right\rangle \gtrless 0$$
$$\Longrightarrow \langle P_t(t;x,y)=1,\ P_t(t;x,y)=0\rangle$$

for all t, and λ^2-a.e. (x,y). No conditions on joint match quitting probabilities are imposed for the nongeneric set of matches which are neither mutually agreeable nor disagreeable. As in SS, I may further reduce this to

$$\begin{aligned} \min\Bigg\langle &\int_t^\infty e^{-\int_t^s \beta + \frac{\rho u_{s'}(x) u_{s'}(y)}{m_{s'}(x,y)} ds'} m_s(x,y)/m_t(x,y)(y-\psi_s(x))\,ds, \\ &\int_t^\infty e^{-\int_t^s \beta + \frac{\rho u_{s'}(x) u_{s'}(y)}{m_{s'}(x,y)} ds'} m_s(x,y)/m_t(x,y)(x-\psi_s(y))\,ds \Bigg\rangle \gtrless 0 \qquad (11) \\ &\Longrightarrow \langle \dot{m}_t(x,y) = \rho u_t(x) u_t(y),\ m_t(x,y)=0\rangle \end{aligned}$$

Theorem 1 (Characterization) *A search equilibrium is a triple: a matching measure* $\langle \hat{\mu}_t \rangle$ *whose density satisfies the mutual agreeability condition* (11), *an unemployment measure* $\langle \hat{\nu}_t \rangle$ *obeying* (1), *and a flow value system* $\langle \hat{\psi}_t \rangle$ *satisfying* (9).

Conjecture 1 (Existence) *A search equilibrium exists. One also exists in pure strategies, i.e. for all* t, $\dot{m}_t(x,y) = \rho u_t(x) u_t(y)$ *if* $y \in \mathcal{A}_t(x)$ *and* $m_t(x,y) = 0$ *if not.*

The proof is not yet done, but I am very close. Essentially, the problem is very close to the point where the Fan-Glicksberg Fixed Point Theorem will apply. As for the purification, intuitively, the only matches that can be randomized over are the weakly mutually matches that are not mutually agreeable.

I now rule out a class of particularly nasty search equilibria. Intuitively, the simultaneous quitting en masse of a positive measure of matches at any time (after time-0) represents an incredible coordination problem, arguably accomplished only by means of a public coordinating device. Yet I am explicitly ruling out 'sunspots': The model must evolve deterministically. Still, the formulation of strategies admits a possible dependence on time, and so conceivably this could occur. With this motivation, I shall define a *bootstrap* search equilibrium to be one in which $\langle \hat{\mu}_t \rangle$ is discontinuous at some positive time i.e. $\langle \hat{\mu}_t \rangle \notin C_{\mathcal{M}^s}[0,\infty)$, the space of weak-continuous functions $\mathbb{R}^+ \mapsto \mathcal{M}^s$. This notion clearly only bites out of steady-state.

Conjecture 2 (Continuity) *Bootstrap search equilibria do not exist.*

The name 'bootstrap' ought to be suggestive of the circular logic underlying the discontinuity. This logic goes as follows. Suppose by way of contradiction that at time t^* a positive mass of agents X *did* quit their matches. Then it must be true that the flow values $\psi_t(x)$ of agents $x \in X$ have discontinuously jumped up. Indeed, one is always willing to stay matched if one is earning a strictly positive flow surplus; conversely as the horizon T^* approached, the option value of staying matched must have been quickly vanishing, so that x would not stay matched unless her flow surplus was not too negative. The idea is then to deduce a contradiction, by arguing that one could have done better earlier by adopting the matching rule that is optimal only starting at t^*.

6. PERMANENT EMPLOYMENT

6.1 Preference Monotonicity

I shall hereafter focus on a particular pure strategy search equilibrium with implied acceptance sets $\mathcal{A}_t(x)$ for all types x.

This section shall essentially assume away the technical difficulties posed by partners quitting on an individual by looking for the matches that are never terminated. Without this threat, the problem everyone confronts here is tantamount to the model of Smith (1994): An individual faces a constant flow arrival rate of prizes from a deterministically evolving absolutely continuous 'prize distribution'. In that set up, individuals accept any match yielding a payoff at least their flow values. More generally, a similar argument reveals that any two individuals having the same strategy are comparable, or

Lemma 3 (Monotonicity) *At any time $s \geq t$, if x is willing to match with y, then x is strictly willing to match with $z > y$, so long as $x \in \mathcal{A}_s(y) \iff x \in \mathcal{A}_s(z)$ for all $s \geq t$.*

Under the above conditions, x is not indifferent between (and thus will not mix over) any $z > y$ at any time $s \geq t$ if x is willing to match with either.

6.2 Cross-Sectional Monotonicity: A Poverty of Riches?

The problem of this paper belongs to a general equilibrium environment. As such, individual preferences feed back rather naturally into all unmatched values. To compare values, I shall now define the inverse of $\mathcal{A}_t$ as in Smith (1995a). The *opportunity set* $\Omega_t(x) = \{y \mid x \in \mathcal{A}_t(y)\}$ consists of all individuals y who are willing to match with x at time t. Whereas $\Omega_t(x) \equiv \mathcal{A}_t(x)$ in a TU model, this need not obtain in the NTU context here. Characterizing the equilibrium entails knowing whether *reciprocal monotonicity* obtains: Is $\Omega_t(y) \subseteq \Omega_t(x)$ for all $x > y$.

So consider two agents $y < x$. The following seductive argument is *false*: "Monotonicity implies that anyone who is willing to match with y is strictly willing to match with x, provided x emulates the acceptance strategy of y. So x could assure herself of $\Omega_t(x)$ if she so desires, and achieve an average value $\mathcal{W}_t(x) = \mathcal{W}_t(y)$. Allowing x to behave optimally, we have $\mathcal{W}_t(x) \geq \mathcal{W}_t(y)$ for all t."

Of course the flaw in this logic is that x might not be able to *credibly* commit to employing the acceptance strategy of y. In fact, x could conceivably have such bright prospects in the near future, that she would plan to quit many matches that y would not. This fact being foreseen, it could be the case that no one wishes to match with x. Truly, future riches may beget current poverty. As it turns out, while the search equilibrium of this paper will contain such individuals with great expectations, I shall argue that the sorry conclusion will *never* arises in equilibrium.

6.3 Cross-Sectional Aggregation

I now give a slightly modified spin on the basic logic underlying the perfect segregation result of Smith (1995a): If starting at some time, types x and y have the same *preferences* and always enjoy identical future *opportunities* (those types willing to match with them, $\Omega_\cdot(x)$ and $\Omega_\cdot(y)$), then they must henceforth make the same current and future *choices*. This simple observation establishes

Lemma 4 (Coincident Choices) *Fix $y \neq x$. Suppose that for all $s \geq t$, anyone willing to match with x is also willing to match with y, and vice versa. Then x and y have the same strategy. That is, if $\Omega_s(x) = \Omega_s(y)$ for all $s \geq t$, then $\mathcal{A}_s(y) \equiv \mathcal{A}_s(x)$, $\mathcal{W}_s(y) \equiv \mathcal{W}_s(x)$, and hence $\theta_s(y) = \theta_s(x)$ for all $s \geq t$.*

As transparent as it is, Lemma 4 encapsulates the core insight of this section, and in fact the paper, as it allows me to amalgamate groups of individuals when calculating equilibrium strategies. For this is a context with both single person and cross-sectional dynamics, and this insight allows me to refrain (only temporarily, as it happens) from seriously tackling the two simultaneously.

Call a set $\mathcal{C}_t$ of individuals with the same acceptance strategy a *coincident choice set.* Note that Lemma 4 shows that $\mathcal{C}_t$ cannot contract over time, or $\mathcal{C}_s \subseteq \mathcal{C}_t$ for all $t > s$. Let $\mathcal{A}_t(\mathcal{C}_t)$ denotes the well-defined common acceptance set of the coincident choice set $\mathcal{C}_t$. An especially salient coincident choice set is an *equivalence class* $\mathcal{E}_t$, whose members will only agree to match with other members of $\mathcal{E}_t$ at time t, i.e. for which $\mathcal{A}_t(\mathcal{E}_t) = \mathcal{E}_t$ for all t. Clearly, an equivalence class $\mathcal{E}_t \subseteq (0,1)$ cannot contract over time, or $\mathcal{E}_s \subseteq \mathcal{E}_t$ for all $t > s$, because this property was already demonstrated for coincident choice sets. Observe also that any match by individuals within an equivalence class $\mathcal{E}_t \subseteq (0,1)$ is never quit. I now show that such a notion is not without example.

Theorem 2 (Permanent Employment) *In any search equilibrium, there exists a coincident choice set $\mathcal{C}_t$ that includes $[\rho/(\rho+\beta),1)$. Moreover, $\mathcal{C}_t$ is a convex interval $[\theta_t,1)$, and in fact $\mathcal{C}_t$ is an equivalence class, i.e. θ_t is weakly decreasing.*

Proof: Because

$$\theta_t(x) \leq \mathcal{W}_t(x) < \int_t^\infty \rho e^{-(\beta+\rho)(s-t)} ds = \rho/(\rho+\beta),$$

if individuals in $[\rho/(\rho+\beta),1)$ plan to stay matched *to one another,* such matches are forever mutually agreeable. Conversely, any two types $x, y \geq \rho/(\rho+\beta)$ have a strict disincentive to quit a match with one another because their respective unmatched values are both strictly less than $\rho/(\rho+\beta)$ in *any* search equilibrium. Hence, they are *not permitted* to plan to quit. Search equilibrium neatly precludes any imperfect behaviour in which one party quits knowing the other plans to do so, when it is really in the interest of neither to quit. Simply put, a match *must* continue if it still provides both parties with positive own-surplus.

Now let's push this argument a little further by looking to dynamic programming for intuition. Suggestively, denote by $\theta \in (0,1)$ the unique solution of[23]

$$\theta = \int_0^\infty e^{-\int_0^s \left(\beta + \int_\theta^1 \rho \, \lambda(dy)\right) ds'} \int_\theta^1 \rho y \, \lambda(dy) \, ds = \frac{\int_\theta^1 y \, \lambda(dy)}{(\beta/\rho) + \int_\theta^1 \, \lambda(dy)}$$

By searching further, individuals cannot hope to do better than achieve an expected payoff of θ. So the previous logic applies, and any match by individuals $x, y \geq \theta$ is mutually agreeable and will never be quit.

The above argument can now be reapplied at every moment in continuous time; *however*, as time progresses, the threshold θ must fall because the measures of remaining unmatched high types is obviously falling. This is one place where the choice of quadratic search technology bites. So write $\theta = \theta_t$, and it is possible to solve for the resulting law of motion of $\langle \theta_t \rangle$

$$\theta_t = \int_t^\infty e^{-\int_t^s \left(\beta + \int_{\theta_{s'}}^1 \rho u_{s'}(y) \, \lambda(dy)\right) ds'} \int_{\theta_s}^1 \rho y u_s(y) \, \lambda(dy) \, ds$$

as well as the evolution equation for $\langle u_t(y) | y \geq \theta_t \rangle$ under the presumption that lower types will quit any extant match as soon as $y = \theta_t$. I defer this exercise, as it is the substance of Lemma 5. This exercise establishes that θ_t is a smooth decreasing function of time. Observe that we can rule that the flow value

I must justify two key assumptions. First, must any type x quit any extant match (with $y < \theta_t$, perforce) as soon as $x = \theta_t$, and second, must any $x \geq \theta_t$ always refuse to match with any type $y < \theta_t$? Note that these are one and the same question, by the presumed nature of the strategy set: The one acceptance set determines which ongoing matches to quit and which potential ones to accept. Also observe that such a common strategy certainly constitutes equilibrium behaviour because $\mathcal{W}_t(x) = \theta_t(x) = \theta_t$ for any $x \geq \theta_t$ by construction, in light of the Bellman optimality principle. Namely, whenever a type finds herself above the threshold, she knows that she may match with any type in $[\theta_t, 1)$ without fear of her partner subsequently quitting; she therefore behaves optimally by accepting precisely those types in $[\theta_t, 1)$.

I now turn to the *necessity* of the above two assumptions. To do so, I first remark that θ_t — having been calculated assuming the smallest possible band of mutually agreeable matching $[\theta_t, 1)$ — is certainly an upper bound for the average value $\mathcal{W}_t(y)$ for any $y \in (0,1)$ in *any* search equilibrium. For any other search equilibrium can only result in less choosy behaviour by types in $[\theta_t, 1)$; this will in turn yield a lower mass of unemployment types in $[\theta_t, 1)$, and thus a lower implied value for them too.

---

[23] Note that there is a solution to this equation because both sides are continuous in $\theta \in (0,1)$, and as θ increases from 0, the left hand side (LHS) ranges from 0 to 1 while the RHS ranges from a positive number down to 0. Moreover, the LHS has slope 1, while the derivative of the RHS continuously tends to $E[y - \theta | y \geq \theta] < 1$ as $\rho/\beta \to \infty$. Hence, the solution is unique for large enough ρ or small enough β — that is, when the search frictions are not too bad. I shall maintain such an assumption.

Observe that if $\theta_t(x)$ is monotonically decreasing, then Smith (1994) implies that $\mathcal{W}_t(x) = \theta_t(x)$ for all t. Hence, x never wishes to quit any match, regardless of how she will be treated by her potential partners. Indeed, since her unmatched value is falling, her incentive to stay in the match only grows with time. But in that case, x can credibly commit to stay matched with any $y \geq \theta_t(x)$ and must turn down all $y < \theta_t(x)$. And since $\mathcal{W}_t(y) \leq \theta_t \leq x$, no type y ever wishes to part company with x. So all matches between x and $y \geq \theta_t(x)$ are mutually agreeable forever afterwards. But if $\theta_t(x)$ is monotonically decreasing for all $x \geq \theta_t$, then it must be invariant in x, since any individual can credibly mimic another, and all such types $x \geq \theta_t$ are desirable matches for anyone else. But in that case, we clearly must have $\theta_t(x) = \theta_t$ for all x, and I am done.

So it remains to rule out $\theta_t(x)$ not being monotonic decreasing, for some or all $x \geq \theta_t$. To do so, I shall consider the function $\bar{\theta}_t(x) = \sup_{s \geq t} \theta_s(x)$. Think of this as 'filling in all the valleys' of $\theta_t(x)$. By results in Smith (1994), whenever $\bar{\theta}_t(x) = \theta_t(x)$, I must additionally have $\mathcal{W}_t(x) = \bar{\theta}_t(x) = \theta_t(x)$. Now assume, by way of contradiction, that $\theta_t(x) < \bar{\theta}_t(x)$ in some search equilibrium. Smith (1994) shows that $\theta_t(x)$ is always an upper semicontinuous function of time (and so, $\bar{\theta}_t(x)$ is a continuous function of time). Hence, there are times $t_1 < t_2$ such that $\theta_t(x) < \bar{\theta}_t(x)$ for all $t \in (t_1, t_2)$, with $\theta_{t_1}(x) = \theta_{t_2}(x) = \bar{\theta}_t(x)$. Thus Smith (1994) yields $\mathcal{W}_{t_1}(x) = \mathcal{W}_{t_2}(x)$.

My goal is to argue that irrational behaviour is implied either at t_1 or t_2. For instance, I could suppose by way of contradiction that behaviour is optimal starting at time t_2. In that case, there are uniformly fewer unmatched above $\theta_{t_1}(x) = \theta_{t_2}(x)$ available at t_2 than at t_1, because in the interum, some have matched with one another, and those matches are not quit. It seems implauble then that we do not have $\mathcal{W}_{t_1}(x) > \mathcal{W}_{t_2}(x)$. So let x starting at time t_1 adopt the acceptance strategy which is optimal starting only at time t_2. That is, let her substitute $\hat{\mathcal{A}}_t(x) = \mathcal{A}_{t-t_1+t_2}(x)$ for all $t \geq t_1$. Being a deviation from optimality, this cannot improve $\mathcal{W}_t(x)$ on $[t_1, \infty)$, and thus any match that was credible for her before is still credible.

(to be continued) $\diamond$

Remark. The analysis is made especially difficult due to the following fact. Recall that faced with a prospect of one's partner quitting, we can only say for certain that y will match with x if $x \geq \psi_t(x)$. But I cannot simply conclude that any type y outside the coincident choice set is always willing to match with $x \geq \theta_t$ for that would require showing $\psi_t(y) \leq \theta_t$. But formula (9) will not easily permit such a deduction. Conceivably, $\psi_t(x) > \theta_t$ can occur if future flow values $\psi_t(x)$ are sufficiently small. In words, an individual doesn't care about a high immediate flow payoff if her unmatched value is rapidly falling, and she knows she will soon get jilted. In fact, I later wish to rule out $\psi_t(x) > x$ ever.

6.4 Dynamic Analysis of Permanent Employment

State Variables and Laws of Motion. I now turn to some unresolved issues from the proof of Theorem 2. With a falling threshold θ_t, individuals $x < \theta_t$ will quit any matches precisely at the moment they are absorbed into $\mathfrak{C}_t$. In light of strategic separability, at time t the highest individual only *statically* cares about the *mass* $\hat{u}_t$ of unmatched individuals $x \geq \theta_t$, and the average type $\hat{\mu}(t)$ among unmatched individuals above θ_t. I shall therefore look for an equilibrium threshold path parameterized by just these variables. For analytic ease, however, I shall slightly transform them. Replace $\hat{u}_t$ by

$$u_t \equiv L(\theta_t) + \hat{u}_t = L(\theta_t) + \int_{\theta_t}^{1} u_t(x)\lambda(dx) \tag{12}$$

Note that u_t is not the global unemployment rate (i.e., the overall fraction of all unmatched individuals at time t), but rather is the fraction of all individuals not in permanent matches at time t. Similarly supplant $\hat{\mu}(t)$ by

$$\pi_t \equiv \hat{\mu}(t)[u_t - L(\theta_t)] + \int_0^{\theta_t} x\lambda(dx) = \int_0^{\theta_t} x\lambda(dx) + \int_{\theta_t}^{1} xu_t(x)\lambda(dx) \tag{13}$$

This yields the natural definitions $u_t = \int_0^1 u_t(x)\lambda(dx)$ and $\pi_t = \int_0^1 xu_t(x)\lambda(dx)$, if I make the erroneous assumption that $u_t(x) = 1$ for all $x < \theta_t$. This assumption will not matter if any x quits her match at as soon as $x = \theta_t(x)$, which by assumption holds.

Lemma 5 (Laws of Motion) *If θ_t is monotonically decreasing, then the triple (θ_t, u_t, π_t) is differentiable and satisfies $\pi_0 = \int_0^1 x\lambda(dx)$, $u_0 = 1$, and*

$$\begin{bmatrix} \dot{\theta} \\ \dot{u} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} \beta\theta + \rho(\theta u - \pi - \int_0^\theta (\theta - x)\lambda(dx)) \\ -\rho[u - L(\theta)]^2 \\ -\rho[u - L(\theta)](\pi - \int_0^\theta x\lambda(dx)) \end{bmatrix} \tag{14}$$

Equilibrium Analysis of the Permanent Matching. So long as I can show that Theorem 2 is true, i.e. equivalence class permanent matching arises in any search equilibrium, I can safely analyze the permanent matching dynamics given by Lemma 5 in isolation of the what is happening outside $\mathfrak{C}_t$. For individuals gladly quit their matches the moment they enter $\mathfrak{C}_t$.

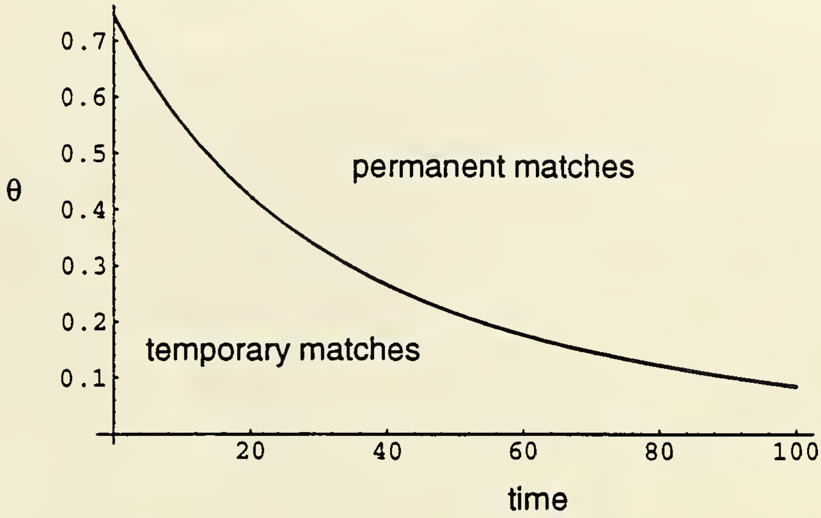
The fixed points of the dynamical system (14) as parameterized by θ are easily found to be

$$\mathfrak{F}_0 = \{(\theta, L(\theta), \int_0^\theta x\lambda(dx))\},$$

But observe that

$$u_t - L(\theta_t) \geq \pi_t - \int_0^{\theta_t} x\lambda(dx) \geq u_t - L(\theta_t)\theta_t \geq 0. \tag{15}$$

Figure 4: **Simulated Equilibrium Threshold Path.** This is drawn for $\beta = 0.01$, $\rho = 1$, and uniform λ (Lebesgue measure). Notice that $\dot{\theta} < 0$ and $\ddot{\theta} > 0$.[26]



Thus, the only *feasible* stationary point of the dynamical system (14) is the 'all-matched' point $(0,0,0)$. Does there exist a path[24] starting at $(\theta, 1, \int_0^\theta x\lambda(dx))$ for some $\theta \in (0,1)$ and tending to $\mathfrak{F}_0$? Note that the nested inequalities in (15) already imply $\dot{\pi} \leq 0$ by economic feasibility, while the laws of motion (14) guarantee $\dot{u} \leq 0$.

Theorem 3 (Existence and Uniqueness) *For any search equilibrium, there exists a unique pattern of permanent matching, with an equivalence class* $\mathcal{C}_t = [\theta_t, 1)$ *for all* $t \geq 0$, *where* $0 < \theta_0 < \rho/(\rho + \beta)$ *and* $\theta_t \downarrow 0$. *The corresponding dynamical system is saddle point stable.*

A depiction of a typical threshold path is found in figure 4. The existence argument is of sufficient interest that it soon follows. I omit the uniqueness proof, but it will follow from a careful examination of the system (14) in light of the constructive existence argument. The saddle-point character of the equilibrium will also follow from that argument, and the Jacobian calculation performed in the proof of Lemma 7.

REMARK. In this outcome, types in $[\theta_t, 1)$ only look for permanent matches with one another. Thus, if $supp(\lambda) \subset (\varepsilon, 1)$, then within finite time, all individuals belong to a single equivalence class, so that all proposed matches are agreed to. By the same token, for any fixed $\varepsilon > 0$, there exists $\rho > 0$ small enough that all proposed matches are agreed to from the outset. Thus, the most of the interesting case is precisely the one I am focusing on, with $\varepsilon = 0$.

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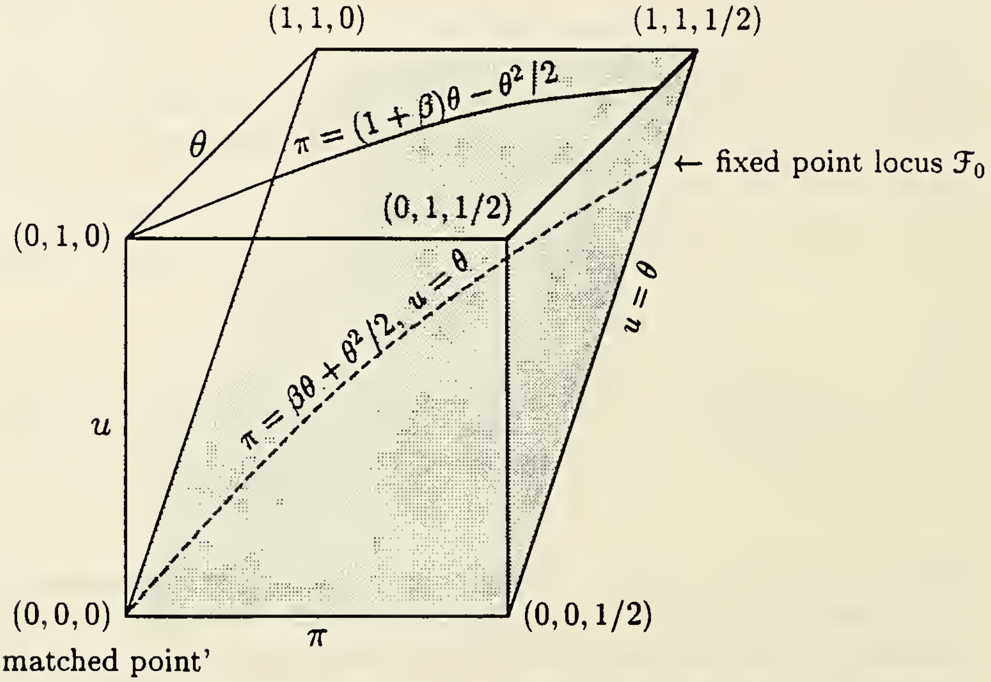
[24] The system (5) *can* by brute force be reduced to an ugly third order non-linear differential equation in θ alone. This is an unprofitable method of analysis.

Figure 5: **System Dynamics in (θ, u, π)-space.** The diagram depicts the dynamics of (14) for $\rho = 1$ and uniform λ, i.e. Lebesgue measure.


(1,1,0)
(1,1,1/2)
$\pi = (1+\beta)\theta - \theta^2/2$
θ
← fixed point locus $\mathcal{F}_0$
(0,1,1/2)
(0,1,0)
$u = \theta$
$\pi = \beta\theta + \theta^2/2, u = \theta$
u
(0,0,0)
'all matched point'
π
(0,0,1/2)

6.5 Beyond the Phase Plane Diagram: An Existence Proof in $\mathbb{R}^3$

A wealth of results in dynamical systems theory is specific to two-dimensional models (eg. the Poincaré-Bendixson Theorem). Indeed, the staple tool for studying dynamics is the phase plane diagram. Clearly, this is not an option with a three-dimensional system. I can, nonetheless, exhibit a geometrical method of proof just for this case.

Suppose WLOG that λ is uniform Lebesgue measure on $(0,1)$, so that its density satisfies $\ell(x) = x$. Consider the triangular 'wedge' in figure 5 defined by $0 \leq \theta \leq 1, 0 \leq u \leq \theta, 0 \leq \pi \leq 1/2$. Let E be the initial 'edge' of the wedge defined by $\pi = 1/2$ and $u = 1$, i.e. no where one is yet matched (shaded dark in Figure 5). I shall show that for the above dynamical system, there exists a unique *economically feasible* path starting on E that eventually hits the origin $(\theta, u, \pi) = (0,0,0)$. I should remark that for generic flows through the wedge, such an occurrence would be truly miraculous (i.e. that a path from a given line intersects a given vertex is a zero probability event)! Nonetheless, I demonstrate that just such a miracle occurs.

Thinking in $\mathbb{R}^3$ is hard. The idea is to reduce the problem to a planar one by focusing on the *faces* F of the wedge. That is, I now show that the locus of 'exits' from the wedge of the dynamical system starting on E passes through the origin.

Lemma 6 *Let Π be the set of all paths starting on the upper edge E of the wedge having $u = 0, \pi = 1/2$. Let $\bar{\Pi} = \Pi \cap F$ denote the locus of exit points of Π from the wedge. Then* $(0,0,0) \in \bar{\Pi}$.

I prove the Lemma in the appendix, but will give a sketch here. Consider a purely geometric exercise that would work if $\bar{\Pi}$ were continuous. Denote a path that starts at $(\theta, 1, 1/2) \in E$ as $x(\theta)$. It's easy to see that for θ_0 near 1, $\dot{\theta}(0)$ is just below $\beta > 0$, and so $x(\theta_0)$ hits F close to *but below* $(1, 1, 1/2)$ on the "sloping face" (see figure 5) of the wedge; similarly, for θ_0 near 0, $\dot{\theta}_0$ is just above $-1/2$, and so $x(\theta_0)$ hits F close to *but below* $(0, 1, 1/2)$ on the "front face" of the wedge.

Put $\rho = 1$. All paths of the autonomous system (5) must exit the wedge on the top face $u = 1$ where $\pi < (1+\beta)\theta - \theta^2/2$, or anywhere on the front face $\theta = 0$, or the lower sloping face $u = \theta$ where $\pi < \beta\theta + \theta^2/2$. Hence, $\bar{\Pi}$ is confined to these faces, and so $x(\theta_0) = (0,0,0)$ for some $0 < \theta_0 < 1$ by continuity of $\bar{\Pi}$.

6.6 Rate of Convergence to Equilibrium

The search frictions obviously prevent the market from clearing immediately. It turns out that the convergence to equilibrium with the quadratic search technology is *very* slow: Rather than than a (negative) exponential convergence rate, convergence occurs at a hyperbolic rate.

Lemma 7 *In the unique equilibrium, the fraction of workers not in permanent matches satisfies* $u_t = o(1/t)$.

The proof of the following result is in the appendix. In a word, the slow convergence is a simple consequence of the fact that no eigenvalue of the associated dynamical system is negative at the origin. The *economic* intuition is equally simple: With a quadratic matching technology, even if individuals accept all matches, the flow matching rate is proportional to the square of the mass of unmatched agents. That is, I can hope for no better (by an order of magnitude) than the above rate of matches.[27] The message of Lemma 7 is that I can in fact asymptotically achieve this best. The appendix also contains an intuitive rationale for this result.

6.7 Comparative Statics

First note that a rescaling of (β, ρ) only serves to renormalize time; therefore, it suffices to investigate the comparative statics in β alone, for instance. Of prime interest is what happens to the unique initial threshold θ_0 as β changes, and especially for β close to 0. Unfortunately, a closed theoretical answer is outside the realm of known analytic methods.[28] Still, calculations on *Mathematica* strongly suggest (see Figure 6) that the first best is approached as search frictions vanish.

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[27] A similar argument establishes that with any matching technology with hazard rate $h(u) = \rho u^\gamma$, some $\gamma \in (0,1]$, the fastest that u_t can decline is if $\mathcal{C}_t = (0,1)$. In that case, $\dot{u} = -u^{\gamma+1}$. yielding $u_t = [(1+\gamma)t+1]^{-1/\gamma}$ — a hyperbolic and not exponential rate. Only with the linear technology, i.e. $\gamma = 0$, is an exponential rate of convergence possible.

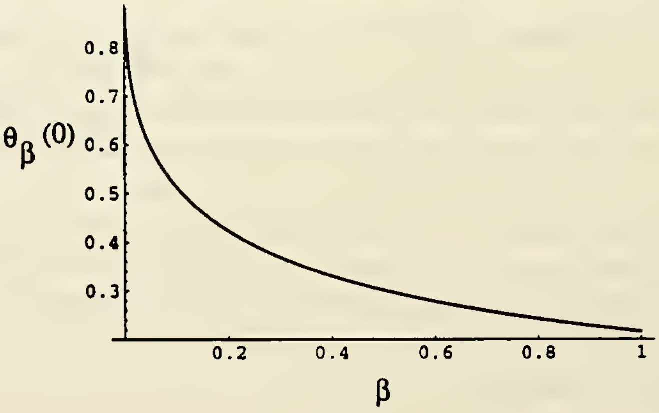
Conjecture 3 *The initial threshold* θ_0 *is monotonic decreasing in the* β/ρ*, with* $\lim_{\beta/\rho\to 0}\theta_0(\beta/\rho) = 1$.

I wish to briefly make the case that this yields a rather unintuitive prediction. Consider the world without any *explicit* source of time discounting, or $\beta = 0$. It is not implausible[29] that the declining quality level in the pool is would provide enough *implicit* discounting to secure an equilibrium. It does not. Using the machinery of this section, one can show that there can exist no search equilibrium.

As an application, the *undiscounted* GNP must approach the frictionless-best $\int_0^1 x^2\lambda(dx)$ as β/ρ vanishes. Indeed, if $\lim_{\beta\to 0}\theta_0(\beta) = 1$ near $u = 1, \pi = 1/2$, then (5) implies that $\lim_{\beta\to 0}\theta'_\beta(0) = 0$. That is, the threshold descends very slowly, so that most matches involve individuals who are arbitrarily close to one another. It is also likely that the *present value* of GNP tends to $\int_0^1 x^2\lambda(dx)$, so that the welfare loss due to search frictions evaporates, but that is harder to show.

It is easy to bound $\theta_0(\beta)$ away from 1, and thus provide a lower limit on the welfare loss (ignoring the welfare contribution made by temporary employment). Supposing that $\ell(x) \equiv x$, I must have $0 < -2\dot{\theta}_0(\beta) = \rho\theta_0(\beta)^2 - 2(\rho+\beta)\theta_0(\beta) + \rho$, which implies that $\theta_0(\beta) < 1 + \beta - \sqrt{\beta^2 + 2\beta}$. Note that the right hand side tends to 1 as $\beta \downarrow 0$. This bound concurs with figure 4, since $\theta_0(.01) < .865$.

Figure 6: **Simulated Initial Thresholds as a Function of** β **when** $\rho = 1$.



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[28] Absent closed form solutions, I can only really describe the local behaviour of 'stable manifolds' *near a fixed point* i.e. *not* near $u = 1, \pi = \int_0^1 x\lambda(dx)$. For instance, manipulation of system (5) reveals that $\ddot{\theta} = \beta\dot{\theta} + \beta\rho\theta(u - L(\theta))$. With $\ell(x) = x$, this reduces to

$$\begin{aligned}\ddot{\theta}_0 &= \beta^2\theta_0 - \beta\rho[1-\theta_0]^2/2 + \beta\rho\theta_0[1-\theta_0]\\ &= (\beta/2)[2\beta\theta_0 - (1-\theta_0)(1-3\theta_0)]\end{aligned}$$

So if (as conjectured) $\ddot{\theta}_0 > 0$, I can still only weakly conclude that $\theta_0 > 1/3$ even when $\beta \to 0$.

[29] That is, I at first believed it.

7. TEMPORARY EMPLOYMENT

The analytical power of the identical vNM preference assumption now comes to the fore as I shall henceforth simply assume the permanent matching of the last section is given. This shall serve as an anchor for all temporary matching decisions. Now one could imagine a model whereby temporary matching tautologically could not arise. Namely, suppose that xy were a one-shot only utility 'flash' instead of the *flow* output of the match (x, y). Then the analysis of the last section would be a complete characterization of the search equilibrium — that is, of who ultimately matches with whom. And in fact, the incomplete proof of Theorem 2 could be very quickly finished.

With the maintained assumption of flow payoffs, individuals awaiting their eventual permanent match will desire to profitably use the interim time period. Given that time is of the essence (as $\beta > 0$), those types below $\mathcal{C}_t$ match early, and later quit at least when they are absorbed in $\mathcal{C}_t$. This will play out in a divergence between their average and flow unmatched values. The analysis of this section is inspired by the discrete time examples; however, it so happens that the effects I have shall describe really only surface in a four or more period model. Below the threshold, the analysis becomes much richer and more complicated to flesh out. Because $\dot{\theta}_t < 0$, different types have distinct absorption times; consequently, there are no equivalence classes below $\mathcal{C}_t$. But more to the point for this section, *every such match dissolution (for this reason) of x with $y \neq x$ is non-mutual.* Whether quits arise for other reasons is something I wish to explore, as it is is a crucial determinant of the nature of the unemployment dynamics.

Intertemporal Effects. Let $\theta^{-1}(x)$ denote the time that $\theta_t = x$, with $\theta^{-1}(x) = 0$ if $x > \theta_0$. Now, an example in Smith (1994) shows that *the threshold partner $\theta_t(x)$ vanishes as $t \uparrow \theta^{-1}(x)$.* But Smith (1994) also proves that for fixed x, the threshold partner $\theta_t(x)$ is an upper semicontinuous functions of time t. Thus, for any $x < \theta_t$, $\theta_t(x)$ is decreasing for t close enough to $\theta^{-1}(x)$.

With the following result, this can also be extended to flow values.

Lemma 8 (u.s.c. Flow Values) *For fixed x, the flow value $\psi_t(x)$ is an upper semicontinuous functions of time t.*

Proof: Look at formula (9). Since $Q_t(s; x, y)$ satisfies the Bayes-Markov property, so does $P_t(s; x, y)$. But this implies that $P_t(s; x, y)$ is upper semicontinuous in t: In other words, for fixed s, $P_t(s; x, y)$ can jump up, but not down in t. Intuitively, since the quitting rule is memoryless, matches begun at close by times cannot be treated capriciously differently, unless the first match is simply unprofitable and is thus not okayed (i.e. forgotten about entirely). Moreover, the search dynamics constrain the unemployment rate $u_t(x)$ to be upper semicontinuous at all x, since it is never the case that a positive mass of any type 'suddenly' finds a match. After integrating and multiplying, the resulting expression for $\psi_t(x)$ is also upper semicontinuous in t. ◇

I believe I can use equation (6) to show that the flow value $\psi_t(x)$ also vanishes as we approach time $\theta^{-1}(x)$. Together with upper semicontinuity from Lemma 8, this implies that for any $x < \theta_t$, the flow value $\psi_t(x)$ is decreasing for t close enough to $\theta^{-1}(x)$. But since $\psi_t(x) = \mathcal{W}_t(x) - \dot{\mathcal{W}}_t(x)/\beta$, I must also have $\dot{\mathcal{W}}_t(x) > 0$ then too. Altogether, I have

Lemma 9 (Intertemporal Monotonicity) *For all $x < \theta_t$, both the flow value $\psi_t(x)$ and threshold partner $\theta_t(x)$ are continuously decreasing in t and eventually vanishing $t = \theta^{-1}(x)$, during which time the average value $\mathcal{W}_t(x)$ is increasing in t, tending towards x.*

Recall that any individual $x < \theta_t$ will quit any match at time $\theta^{-1}(x)$. But by Lemma 9, if a match with y ever makes sense for x, then it will continue to provide flow surplus until time $\theta^{-1}(x)$. Hence,

Corollary (Once in a Lifetime Quits) *Any $x < \theta_t$ never quits a match before time $\theta^{-1}(x)$. As a result, she only quits a match at most once in her lifetime.*

Observe in passing that with self-preference, if x ever does match with x, then that match is permanent.

As both a result in and of itself, and a key tool in the analysis that follows, I would like to be able to make the following conclusion about optimal strategies.

Lemma 10 (Monotonic Preferences) *For each $x < \theta_t$, $\mathcal{A}_t(x) = [\theta_t(x), 1)$, for some $\theta_t(x) \in (0, \theta_t)$.*

Let me try to develop the intuition for this one by simply determining the acceptance set of a given individual. To this end, let $\theta_t(x|s)$ be the minimum type with whom x will agree to match at time t whom x knows will quit at time $s > t$. In other words, it satisfies the dynamic arbitrage equation

$$\mathcal{W}_t(x) \equiv (1 - e^{-\beta(s-t)})\theta_t(x|s) + e^{-\beta(s-t)}\mathcal{W}_s(x)$$

Think of the threshold partner as coming with an eternal option, whereas $\theta_t(x|s)$ is more akin to the American options which can be exercised at any moment in some finite time period. This quantity is intuitively sandwiched between the flow value and threshold partner of x, or $\theta_t(x) \leq \theta_t(x|s) \leq \psi_t(x)$. Lemma 10 is making the assertion that $\theta_t(x|\theta^{-1}(y)) < y$ for all $y \geq \theta_t(x)$. In other words, the higher flow reward from x matching with higher types y more than compensates for the shorter period with which they are planning to remain matched with x.

One approach to proving the above result is rather blunt. Any type x is willing to match with any $y \geq \psi_t(x)$. So eventually $\psi_t(x) < x$, and individuals are willing to match with 'themselves'. It would clearly suffice to establish that this *self-preference* holds from the outset for everyone. Since the value of type x starts at $\mathcal{W}_t(x) = x$ and proceeds to decrease the moment she is absorbed by $\mathcal{C}_t$, this is not at all implausible. On the other hand, low enough types can hope to temporarily match with very high types early on, and so $\psi_t(x)$ could potentially be very high. The following lemma resolves the debate.

Conjecture 4 (Self-Preference) *For any t and x, the flow value $\theta_t(x) \leq x$ for all $x < \theta_t$. Thus, $[x, 1) \subseteq \mathcal{A}_t(x)$, or in particular x is always willing to match with another x.*

My idea for the proof is to proceed by contradiction, and focus on the last time that $\psi_t(x)$ continuously slices through x from above. I haven't yet managed to properly formulate this.

Note that self-preference in particular rules out the discrete time 'Groucho Marx' effect. It must purely stem from the lumpiness of discrete time.

Observe that no two distinct types have the same future opportunities, as their date of absorption into the top equivalence class is different. It is now within my grasp to establish a result that seemed out of reach in section 6.

Lemma 11 (Value Monotonicity) *For any t, if $y < x$, then $\mathcal{W}_t(x) \leq \mathcal{W}_t(y)$ with equality exactly when $x > \theta_t$.*

Proof: Any $y > x$ can mimic the acceptance strategy of x and assure herself of x. This is a credible offer because her flow value if falling, by Lemma 9. But y strictly prefers not to mimic x starting at time $\theta^{-1}(y)$. Hence, $\mathcal{W}_t(y) > \mathcal{W}_t(x)$. ◇

Theorem 4 (Portmanteau Characterization) *Any $x < \theta_0$ has a threshold path $\langle \theta_t(x) \rangle$ that is differentiable and strictly monotonic decreasing, and satisfies $\theta_0(x) \in (0, x)$, $\theta_t(x) \to 0$ as $t \uparrow \theta^{-1}(x)$, and $\theta_{\theta^{-1}(x)-}(x) = 0$.*

This theorem is partially summarized in Figure 7.

Figure 7: **Conjectured Temporary Employment Dynamics.** I claim that every type y's threshold partner lies below that type, and monotonically decreases to 0 as $t \downarrow \theta^{-1}(y)$. The schematic diagram is drawn for very low search frictions, so that θ_0 is close to 1.

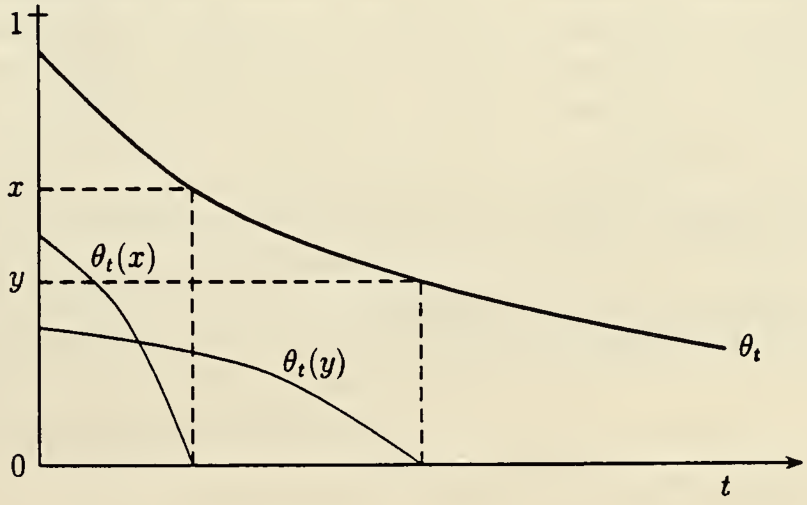
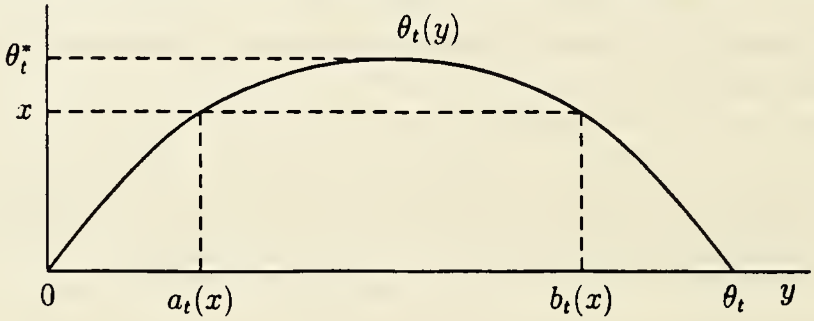


Figure 8: **Conjectured Cross-Sectional Temporary Matching.** Individuals somewhere between 0 and θ_t have the highest threshold partner. For individuals $x < \theta_t^*$, this renders many individuals opportunity sets the union of two intervals, namely, the very high $[b_t(x), \theta_t)$ and very low $(0, a_t(x))$ types.



Cross-Sectional Effects. By Lemma 9, individuals very close to but just below θ_t have very low flow values. Moreover, very low individuals with poor opportunity sets also intuitively have low flow values. On the other hand, the highest flow value ought to be bounded below θ_t. This suggests the following basic cross-sectional result:

Conjecture 5 (Opportunity Sets) *Let* $\theta_t^* = \max_{x \in (0,\theta_t)} \theta_t(x)$. *Then* $\Omega_t(x) = (0, \theta_t)$ *for any* $x \in [\theta_t^*, \theta_t)$. *For any* $x \in (0, \theta_t^*)$, *there exists* $0 < a_t(x) < b_t(x) < \theta_t$ *such that* $\Omega_t(x) = (0, a_t(x) \cup [b_t(x), \theta_t)$.

This result is partially summarized in Figure 8. Note that if self-preference does not obtain at time t, then it must be violated by the individual with the highest flow value, i.e. that $y < \theta_t$ for whom $\theta_t(y) = \theta_t^*$.

Note that even with all the basic theory I have posited, with a separate threshold function $\theta_t(x)$ for all types $x < \theta_t$, the resulting model only admits resolution via *partial* differential equations. Thus, the identical vNM assumption effectively renders the analysis for much of the model an exercise in differential equations instead.

There is one other property worthy of note that follows from the results and conjectures of this section.

Conjecture 6 (Single Crossing Property) *If* $x < y < \theta_t$ *and* $\theta_t(y) = \theta_t(x)$, *then* $\dot{\theta}_t(y) < \dot{\theta}_t(x)$.

This was apparent in Figure 7, for instance.

8. CONCLUDING REMARKS

This paper has capitalized on the identical preferences assumption, the analysis of temporary matching has not benefited from this restriction. This is not intended to be a general theory of matching in NTU search games, which will have to wait. The extension to a model with two fundamentally different agents (such as firms and workers) is not hard, but neither does the exercise add much to the understanding of the issues raised here. Furthermore, the (standard) addition of an exogenous flow of match dissolutions doesn't add much, but greatly complicates life. Essentially, the dynamics will tend to the steady-state identified in Smith (1995a). Namely, there will in the limit be a unique equilibrium with infinitely many equivalence classes of agents willing to match with one another. Finally, moving to the more standard TU world is the subject of SS.

A. APPENDIX: OMITTED PROOFS AND DISCUSSIONS

A.1 Perfect Segregation in NTU Search Models

Smith (1995a) effectively shows that perfect segregation arises in any general NTU search in which all individuals have the same vNM preferences over all potential matches and the unmatched state. What this means is that any the analysis of this paper applies for any symmetric production function $f(x,y) = g(x)g(y)$, where $g' > 0$, and search costs $c(x) = \bar{c}g(x)$ are 'proportional'. For instance, the production function[30] $f(x,y) = x + y$ is not in this class, because any two individuals $x \neq y$ have inequivalent vNM preferences: An arbitrary lottery over $\{\langle x+z, 0 < z < 1\rangle, 0\}$ is not a positive affine transformation of the same lottery over $\{\langle y+z, 0 < z < 1\rangle, 0\}$.

The analysis of this paper, which is derived assuming $g(x) \equiv x$ and $\bar{c} = 0$, also suffices for all search models with $f(x,y) = g(x)g(y)$ and $\bar{c} > 0$. To see how to handle this more general environment, assume first for simplicity that $\bar{c} = 0$. Observe that the substitution $X = g(x)$ reduces this separable production technology to $\xi(X,Y) = XY$, but with L replaced by $L \circ g^{-1}$. Indeed, just verify that $\nu(a \,|\, x(\iota) \leq m) = \nu(a \,|\, x(\iota) \leq g^{-1}(m)) = L(g^{-1}(m))$.

Next consider positive search costs. With $\bar{c} > 0$ (and $g(x) = x$, for simplicity) individuals $x < \bar{c}$ will drop out of the model; the remaining individuals' behaviour will remain unchanged if one renormalizes their search costs to 0, and adds $\bar{c}$ to all remaining searchers' indices. The latter change is formally equivalent to a different type distribution $\hat{\ell}(x) = \ell(x+\bar{c})$. Only the case of 'unemployment insurance' $\bar{c} < 0$ is not (immediately) soluble in what follows!

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[30] This is reminiscent of Lockwood (1986).

A.2 Derivation of Discrete Time Thresholds θ_1 and θ_2

For the first paradigm, the arbitrage definitions of θ_1 and θ_2 yield

$$\theta_1 = \delta\frac{(1-\theta_1)\theta_1(\theta_1+1)/2 + (\theta_1-\theta_2)(\theta_1+\theta_2)/2 + (1-\theta_2)\theta_2^2}{(1-\theta_1)\theta_1 + (\theta_1-\theta_2) + (1-\theta_2)\theta_2}$$

and

$$\theta_2 = \delta\frac{(1-\theta_1)\ell\theta_1(\theta_1+1)/2 + (\theta_1-\theta_2)\ell(\theta_1+\theta_2)/2 + \theta_2^2/2}{(1-\theta_1)\ell\theta_1 + (\theta_1-\theta_2)\ell + \theta_2},$$

where

$$\ell = \frac{(1-\theta_1)\theta_1}{(1-\theta_1)\theta_1 + (\theta_1-\theta_2) + (1-\theta_2)\theta_2} = \frac{(1-\theta_1)\theta_1}{2\theta_1 - \theta_1^2 - \theta_2^2}.$$

For the second paradigm, the denominator is omitted from the equation for θ_1. In either case, there is a unique solution of the form $0 < \theta_2 < \theta_1 < 0$ to these equations.

A.3 Do Markovian Quitting Rules Capture All Mixed Strategies?

For an individual, let $\mathcal{Q}$ be the space of (measurable) Markovian quitting rules $Q : \mathbb{R}_+ \times \Sigma \to [0,1]$ for fixed x. Putting aside any problems wrought by continuous time, Kuhn's Theorem (see Fudenberg and Tirole (1991)) need not apply when strategies are not fully conditioned on history; i.e., there is no reason to expect that for each player's mixture over pure acceptance sets (determined by a time-0 randomization), there is a strategically equivalent *behavioural* strategy (in which one randomizes as one proceeds). That is, $\mathcal{Q}$ is isomorphic to a *strict* subset of $\Delta(\mathcal{T})$. Indeed, let $\omega_{t_1t_2}$ denote the pure strategy, "Be willing to match with y at all times in $[0,t_1)$ and $[t_2,\infty)$." Then there is no way to represent the mixed strategy "equal chance of σ_{13} and σ_{24}" as a Markovian quitting rule, because it isn't memoryless, while the above mixture requires that time-2 behaviour depend on whether a quit obtained at time-1.

A.4 Caveat: Independence Difficulties with a Continuum of Agents

There is a standard abuse of the law of large numbers that this paper commits.[31] Essentially, each individual is assumed to *behave as if* the agents to which she can expect to be matched are (independent) draws from a *probability* distribution equal to the current aggregate *population* distribution. Patch-ups exist for this problem.

But given the matching-theoretic nature of this paper, such annoying difficulties here are subtler still, as the above independence assumption must hold not only for one individual facing a given distribution, but must be jointly true of all individuals each facing an endogenous distribution (jointly determined by the players' actions). The frontier of the literature has not yet been pushed this far.[32]

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[31] Judd (1985) and Feldman and Gilles (1985) codiscovered the problem.

Yet such annoying complications capture absolutely no economics, and have not forestalled progress in search theory. Just as others have done, I shall simply make plausible assumptions that are reasonable in the corresponding totally discretized model. In this way, the nonatomicity assumptions will only serve to facilitate

A.5 Proof of Lemma 5

I first note the following useful 'folk result' on hazard rates.

Lemma 12 (Hazard Rates) *Consider an event which has a piecewise continuous flow probability $h(t)$ of occurring at time t. The chance that it does not occur by time t equals $P(t) = e^{-\rho \int_0^t h(s)ds}$.*

I have never seen an explicit statement or proof of this simple result. To see why it is true, observe that $P(0) = 1$ and, for small $\varepsilon > 0$, $P(t+\varepsilon) = P(t)[1 - \varepsilon h(t)]$. Rearranging terms, and letting $\varepsilon \downarrow 0$, I have $\dot{P}(t)/P(t) = -h(t)$ a.e. This integrates to $ln\, P(t) = -\int_0^t h(s)ds$, yielding the desired result.

Now let's return to establishing the equations of Lemma 5, which are central the paper. I am able to use the simplification $V_t = \theta_t$ since θ is nondecreasing. With no discounting, θ_t is an optimal threshold for the "eligible" individuals $x \geq \theta_t$ if the alternative of waiting provides the same expected *eventual* partner. But for anyone eligible, any (not necessarily acceptable) match proposed at time t provides an expected eventual partner's index of

$$\begin{aligned} \hat{\mu}_t(1 - L(\theta_t)/u_t) + \theta_t(L(\theta_t)/u_t) &= (\pi_t + \int_0^{\theta_t} (\theta_t - x)\ell(x)dx)/u_t \\ &= \theta_t + \int_{\theta_t}^1 (x - \theta_t)\ell(x)u_t(x)dx \end{aligned}$$

This expression reflects the $L(\theta_t)/u_t$ chance that the match will be rejected, plus the fact that $\hat{\mu}_t \equiv [\pi_t - \int_0^{\theta_t} x\ell(x)dx]/[u_t - L(\theta_t)]$. Integrating with respect to the

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[32] The most closely related work is that of Boylan (1992a), who obtains positive results for countably-infinite populations in discrete time. Boylan (1992b) obtains some results in continuous time. See also Gilboa and Matsui (1992) for a general but unrelated model of matching.

"next arrival density" implicit in Lemma 12, I discover[33]

$$\begin{aligned}\theta_t &= \int_t^\infty \rho u_s e^{-\int_t^s \rho u_r + \beta dr} (\pi_s + \int_0^{\theta_s} (\theta_s - x)\ell(x)dx)/u_s ds \\ &= \int_t^\infty \rho e^{-\int_t^s \rho u_r + \beta dr} \left(\pi_s + \int_0^{\theta_s} (\theta_s - x)\ell(x)dx \right) ds\end{aligned}$$

By the Fundamental Theorem of Calculus, θ_t is differentiable, and so the chain rule yields the desired *optimality equation*:

$$\dot{\theta}_t = (\beta + \rho u_t)\theta_t - \rho(\pi_t + \int_0^{\theta_t} (\theta_t - x)\ell(x)dx) \tag{16}$$

Next notice that the flow probability that a given individual x will meet an acceptable partner $x \geq \theta_t$ at time t equals $\rho[u_t - L(\theta_t)]$. Hence,

$$\dot{u}_t(x) = \begin{cases} -\rho[u_t - L(\theta_t)]u_t(x) & \text{for } x \geq \theta_t \\ 0 & \text{for } x < \theta_t \end{cases} \tag{17}$$

if there are no quits. Because individual x quits any other matches by time $\theta^{-1}(x)$, $u_t(\theta_t) \equiv 1$. Differentiation of (12) is also permitted, yielding:

$$\begin{aligned}\dot{u}_t &= \dot{\theta}_t L(\theta_t) + \int_{\theta_t}^1 \dot{u}_t(x)\ell(x)dx - \dot{\theta}_t L(\theta_t) \\ &= -\int_{\theta_t}^1 \rho[u_t - L(\theta_t)]u_t(x)\ell(x)dx \\ &= -\rho[u_t - L(\theta_t)]^2\end{aligned}$$

Similarly, application of (17) to (13) yields the law of motion for π_t.

$$\begin{aligned}\dot{\pi}_t &= \theta_t\dot{\theta}_t L(\theta_t) + \int_{\theta_t}^1 x\dot{u}_t(x)\ell(x)dx - \theta_t\dot{\theta}_t L(\theta_t) \\ &= -\int_{\theta_t}^1 x\rho[u_t - L(\theta_t)]u_t(x)\ell(x)dx \\ &= -\rho[\pi - \int_0^{\theta_t} x\ell(x)dx][u_t - L(\theta_t)]\end{aligned}$$

This completes the proof.

---

[33] As a check on the analysis, and perhaps as a more intuitive approach for some, I can also derive the optimality equation (16) by choosing the integrand to be the value of an *acceptable* match, and integrating with respect to the density of the next *acceptable* arrival:

$$\begin{aligned}\theta_t &= \int_t^\infty \rho[u_s - L(\theta_s)]e^{-\int_t^s \rho[u_r - \theta_r] + \beta dr}[\pi_s - \int_0^{\theta_s} x\ell(x)dx]/[u_s - L(\theta_s)]ds \\ &= \int_t^\infty \rho e^{-\int_t^s \rho[u_r - \theta_r] + \beta dr}[\pi_s - \int_0^{\theta_s} x\ell(x)dx]ds\end{aligned}$$

Now just differentiate.

A.6 Proof of Lemma 6

As noted, $(0,0,0)$ is the only feasible stationary point of the dynamical system (6.4), and in fact the only point of $\mathcal{F}_0$ inside the 'wedge'. Thus, there are no sinks inside the wedge. Each portion of the dynamical system (5), plus the join between the two, is infinitely-differentiable. It is an easy exercise to demonstrate that the resulting system must be Lipschitz. It follows by standard results (eg. Hirsch and Smale (1974)) that the dynamical system is continuous in initial conditions after any *fixed finite time interval.*

It is then possible to show that if the path $x(\theta)$ exits the wedge in some boundedly finite time $\tau(\theta) < \bar{\tau}$ for $\theta < \bar{\theta}$, then the locus $\bar{\Pi}$ of exit points is continuous on F. But $\tau(\theta) < \infty$ for some $\theta \in (0,1)$, with $x(\theta)$ exiting on the front face $\theta = 0$ for small θ. By letting $\bar{\tau} \to \infty$, I see that τ *must be continuous whenever it is finite.* Since the only stationary point of the dynamical system on the front face of the wedge is $(0,0,0)$, if $\tau(\theta) \longrightarrow \infty$ as $\theta \longrightarrow \theta'$, then necessarily $x_t(\theta) \uparrow (0,0,0)$, as required.

A.7 Proof of Lemma 7:

INTUITIVE JUSTIFICATION. Consider the following corollary to Lemma 12, which is an analogue of a familiar probability result.[34]

Lemma 13 (Poisson Borel-Cantelli Lemma) *Let a sequence of events A or B occur according to some stationary Poisson process. Let the draws be independent at the realized Poisson times* $t_1, t_2, \ldots$. *Assume that conditional on an event occurring at any time* $t \geq 0$, *that the chance that it is an A is* $g(t)$. *Then A is never observed with positive probability if and only if* $\int_0^\infty g(t)dt < \infty$.

Think of A as the event that an unmatched partner arrives at a potential meeting. It is easy to see that the measure $g(t)$ of C_t must satisfy this inequality. Indeed, the fastest that g can vanish is if all individuals in C_t agree to all matches with one another, i.e. $g'(t) = -\rho g(t)^2$. This yields $g(t) = (\rho t - g_0)^{-1}$, so that clearly $\int_0^\infty g(s)ds = \infty$.

SERIOUS PROOF. As before, first suppose that $L(x) \equiv x$ for simplicity. Consider the fixed points $\mathcal{F}_0$ of the system (5). Next to determine the stability properties of $(0,0,0)$, I shall first linearize everything. That is, write (5) as $(\dot{\theta}, \dot{u}, \dot{\pi}) = D\ell(\theta, u, \pi) \cdot (\theta, u, \pi)$, where

$$D\ell(\theta,u,\pi) = \begin{bmatrix} \beta + u - \theta & \theta & -1 \\ 2(u-\theta) & -2(u-\theta) & 0 \\ \pi + \theta(u - 3\theta/2) & \theta^2/2 - \pi & \theta - u \end{bmatrix}.$$

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[34] As written, the lemma is not quite in standard form. The garden variety Borel-Cantelli Lemma, a bulwark of modern probability theory, states that if $A_1, A_2, \ldots$ is an infinite sequence of independent events, then $Pr[A_i$ infinitely often$]$ is zero or one as $\Sigma_1^\infty Pr[A_i] < \infty$ or $\Sigma_1^\infty Pr[A_i] = \infty$. The result deals with the case where A_k is the event that the individual meets a prize in $\mathcal{S}_k$ at time k.

This immediately yields

$$D\ell(0,0,0) = \begin{bmatrix} \beta & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

The Jacobian matrix (18) shows that near the origin, the linearized system explodes except along the plane $\pi = \beta\theta$. Consequently, in the non-linear system (5), in any equilibrium, I must have near the origin $\pi = \beta\theta + o(\theta)$. Combining this with (15) yields $u \geq \theta + \pi - \theta^2/2 = (1+\beta)\theta + o(\theta)$, so that for t large enough, i.e. for θ small enough,

$$[u_t - L(\theta_t)]/u_t \geq 1 - \theta/[(1+\beta)\theta + o(\theta)] \approx \beta/(1+\beta),$$

for some $\varepsilon > 0$. Given the law of motion (5), (19) yields $\dot{u}_t/u_t^2 \lessdot -\rho(\beta/(1+\beta))^2$ for large enough t. Integration then yields $u_t \gtrdot (1+\beta)^2/(\rho\beta^2 t + \bar{c})$ for some constant $\bar{c} > 1$. On the other hand, (14) easily implies that $\dot{u}_t/u_t^2 > -1\rho$ for all $t \geq 0$, i.e. $u_t < c_1/(\rho t)$ for some constant $c_1 > 1$. As previously alluded to, a higher ρ merely results in faster convergence by a factor $1/\rho$, yielding a lower unemployment rate.

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